CHAPTER

6

Quadratic Equations

Many real-world situations can be modelled using quadratic relations. Related problems can be solved using quadratic equations. For example, designs in architecture and engineering use parabolic arches. Because of constant gravity, the flight path of a ball, a rocket, or a skateboarder is a parabola. In business, quadratic models are used to solve problems of maximizing revenue or minimizing the cost of production.

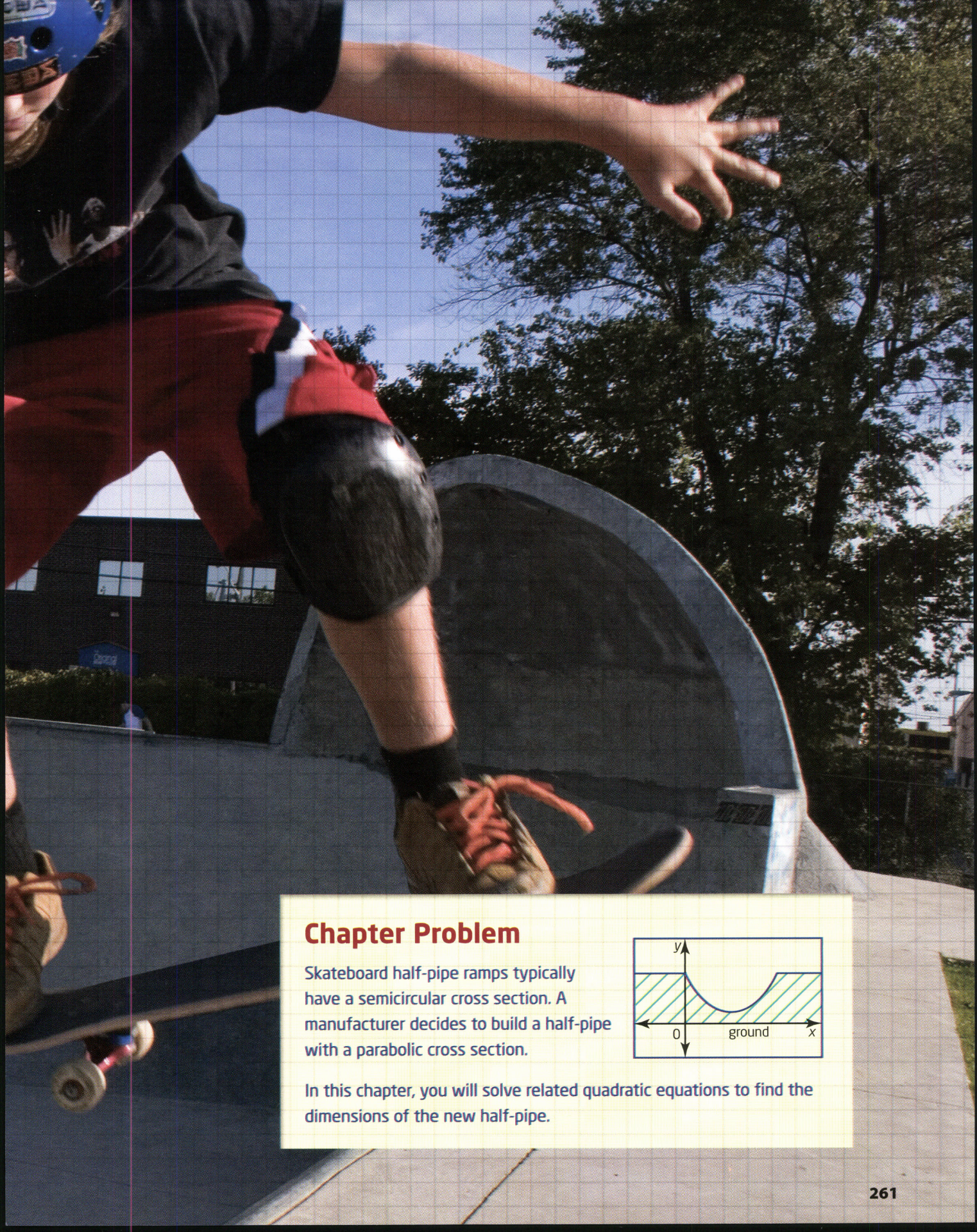
In this chapter, you will relate quadratic equations to the graph of a quadratic relation and solve problems by modelling with quadratic equations.

Quadratic Relations of the Form $y = ax^2 + bx + c$

- Express $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$ by completing the square in situations involving no fractions, using a variety of tools.
- Sketch or graph a quadratic relation whose equation is given in the form $y = ax^2 + bx + c$, using a variety of methods.
- Determine the zeros and the maximum or minimum value of a quadratic relation from its graph or from its defining equation.
- Solve quadratic equations that have real roots, using a variety of methods.
- Determine, through investigation, and describe the connection between the factors of a quadratic expression and the x-intercepts of the graph of the corresponding quadratic relation, expressed in the form $y = a(x - r)(x - s)$.
- Interpret real and non-real roots of quadratic equations, through investigation using graphing technology, and relate the roots to the x-intercepts of the corresponding relations.
- Explore the algebraic development of the quadratic formula.
- Solve problems arising from a realistic situation represented by a graph or an equation of a quadratic relation, with and without the use of technology.

Vocabulary

completing the square
quadratic equation
root
quadratic formula



Chapter Problem

Skateboard half-pipe ramps typically have a semicircular cross section. A manufacturer decides to build a half-pipe with a parabolic cross section.

In this chapter, you will solve related quadratic equations to find the dimensions of the new half-pipe.

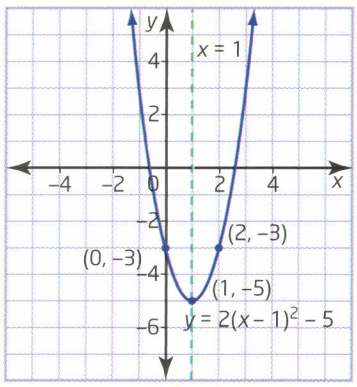
Get Ready

Graph Quadratic Relations of the Form $y = a(x - h)^2 + k$

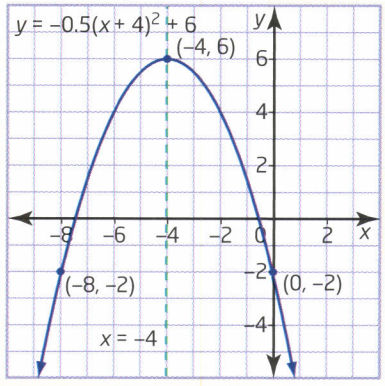
The graph of a **quadratic relation** of the form $y = a(x - h)^2 + k$ is a **parabola** with **vertex** at (h, k). The vertex is a minimum point when $a > 0$ and a maximum point when $a < 0$.

The **axis of symmetry** of the parabola has equation $x = h$.

Substitute $x = 0$ to find the y-intercept.



First, I plot the vertex and draw the axis of symmetry. Then, I determine one point on the curve, for example, the y-intercept, and plot that point. Next, I use symmetry to plot a second point. Finally, I draw a parabola through the points.



1. Sketch each parabola. Label the vertex, the axis of symmetry, and the y-intercept. Use a graphing calculator to check your answers.

a) $y = (x - 4)^2 - 1$

b) $y = 2(x + 3)^2 - 3$

c) $y = -(x + 2)^2 + 5$

d) $y = 3(x - 2)^2 + 2$

2. Sketch each parabola. Label the vertex, the axis of symmetry, and the y-intercept. Use a graphing calculator to check your answers.

a) $y = -0.5(x + 1)^2 + 2$

b) $y = -2(x - 3)^2 + 10$

c) $y = \frac{1}{2}(x - 3)^2 - 7$

d) $y = -\frac{3}{4}(x + 1)^2 + 6$

Square Roots

Since $81 = 9 \times 9$ and $81 = (-9) \times (-9)$, the square roots of 81 are $+9$ and -9.

Using mathematics notation, this can be written as $\pm\sqrt{81} = \pm 9$.

Literacy Connections

The symbol $\pm$ means *positive or negative.*

$\pm\sqrt{81}$ is read as *plus or minus the square root of 81.*

3. Find the square roots of each number, where possible. If necessary, round to the nearest tenth.

a) 100 **b)** 36

c) 75 **d)** -9

4. Use the order of operations to evaluate each expression.

a) $\pm\sqrt{25 - 16}$ **b)** $\pm\sqrt{3^2 + 4^2}$

c) $\pm\sqrt{6^2 - 11}$ **d)** $\pm\sqrt{8^2 - 3(5)}$

Factor Quadratic Expressions

When factoring a **quadratic expression**, first check for any common factor or special patterns.

Quadratic Expression	$x^2 + bx + c$	$a^2 - b^2$	$a^2 + 2ab + b^2$ $a^2 - 2ab + b^2$	$ax^2 + bx + c$
Factoring Technique	Find two integers, r and s, with a product of c and a sum of b. Then, write $x^2 + bx + c$ as $(x + r)(x + s)$.	Use the **difference of squares** pattern. $a^2 - b^2$ $= (a - b)(a + b)$	Use a **perfect square trinomial** pattern. $a^2 + 2ab + b^2 = (a + b)^2$ or $a^2 - 2ab + b^2 = (a - b)^2$	Find two integers with a product of $a \times c$ and sum of b. Then, break up the middle term and factor by grouping.
Example	For $x^2 + 11x + 18$, $b = 11$ and $c = 18$. The two integers are 9 and 2. $x^2 + 11x + 18$ $= (x + 9)(x + 2)$	$100x^2 - 9$ $= (10x)^2 - 3^2$ $= (10x - 3)(10 + 3)$	$x^2 + 6x + 9$ $= x^2 + 2(x)(3) + 3^2$ $= (x + 3)^2$ $25x^2 - 40x + 16$ $= (5x)^2 - 2(5x)(4) + 4^2$ $= (5x - 4)^2$	For $6x^2 - 11x - 7$, $a \times c = -42$ and $b = -11$. The two integers are 3 and -14. $6x^2 - 11x - 7$ $= 6x^2 + 3x - 14x - 7$ $= (6x^2 + 3x) - (14x + 7)$ $= 3x(2x + 1) - 7(2x + 1)$ $= (2x + 1)(3x - 7)$

5. Factor.

a) $2x^2 + 10x + 12$ **b)** $x^2 + 16x + 28$

c) $x^2 - 16x + 63$ **d)** $4x^2 - 36$

e) $81x^2 - 49$ **f)** $3x^2 - 6x - 24$

g) $16x^2 - 8x + 1$ **h)** $12x^2 + 19x + 4$

6. Factor.

a) $x^2 - 4x - 32$ **b)** $2x^2 + 12x + 18$

c) $64x^2 - 1$ **d)** $144x^2 - 312x + 169$

e) $49x^2 + 70x + 25$ **f)** $2x^2 - 9x - 5$

g) $9x^2 - 18x + 8$ **h)** $6x^2 + 21x + 9$

Translate From Words to Algebra

Sentence	Algebraic Expression
One number is double another.	The first number is x. The second number is $2x$.
The length is 6 cm more than the width.	The width is w. The length is $w + 6$.
The regular price of \$5 is reduced by 25¢ an unknown number of times.	The number of price reductions is n. The new price, in cents, is $500 - 25n$.

7. Translate using algebraic expressions. Use one variable to represent the unknowns in each sentence.

a) A number is half another number.

b) The area of the garden is tripled.

c) The regular price of \$12 is reduced by \$1 an unknown number of times.

d) A number is the sum of the squares of two consecutive numbers.

e) The length is three less than double the width.

8. Write an equation to represent each sentence, using two variables.

a) The sum of two numbers is 100.

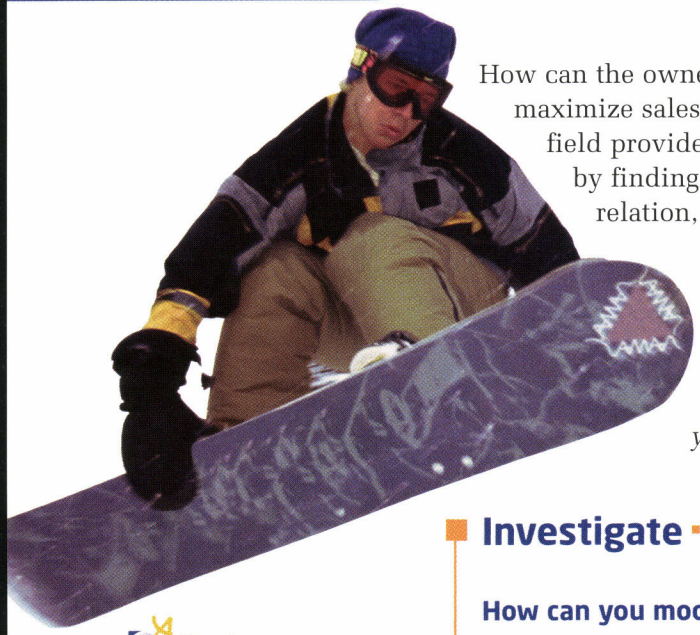
b) The perimeter of the rectangle is 50 cm.

c) Find the perimeter when the length is triple the width.

d) The product of two numbers is equal to the square of double the smaller number.

e) A triangle with a perimeter of 12 cm has one side measuring 3 cm.

6.1 Maxima and Minima



How can the owner of a snowboard rental business use mathematics to maximize sales or minimize cost? What dimensions of a rectangular field provide the greatest area? Questions like these are answered by finding the maximum or minimum point of a quadratic relation, which occurs at the vertex.

If a relation is of the form $y = a(x - h)^2 + k$, then the vertex is (h, k). However, if a relation is of the form $y = ax^2 + bx + c$, the coordinates of the vertex are not so obvious. In this section, you will learn how to express $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$.

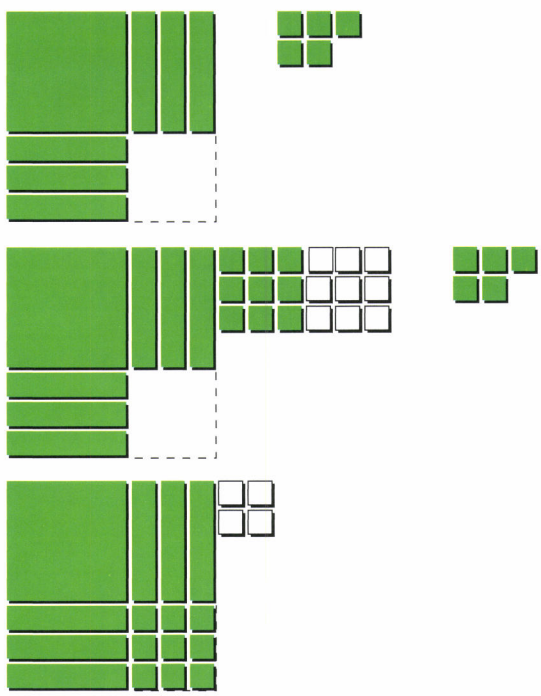
Investigate

Tools

- algebra tiles
- graphing calculator

How can you model the process of creating a perfect square?

1. Consider the quadratic expression $x^2 + 6x + 9$.
   a) Show that the expression is a perfect square using algebra tiles.
   b) Factor the expression as a perfect square.
2. Repeat step 1 using the quadratic expression $x^2 + 4x + 4$.
3. Consider the quadratic expression $x^2 + 6x + 5$.
   a) Describe how algebra tiles have been used to create a perfect square using the *first two terms*.
   b) Explain how the relation $y = x^2 + 6x + 5$ relates to $y = (x + 3)^2 - 4$.
   c) Use a graphing calculator to compare the graphs of $y = x^2 + 6x + 5$ and $y = (x + 3)^2 - 4$.



4. Consider the quadratic expression $x^2 + 4x + 3$.
   a) As in step 3, use algebra tiles or a diagram to create a perfect square using the first two terms.
   b) Explain how the relation $y = x^2 + 4x + 3$ relates to $y = (x + 2)^2 - 1$.
   c) Use a graphing calculator to compare the graphs of $y = x^2 + 4x + 3$ and $y = (x + 2)^2 - 1$.
5. **Reflect** Illustrate and explain how to use algebra tiles to rewrite the quadratic relation $y = x^2 + 2x + 7$ in the form $y = (x - h)^2 + k$.

The process of **completing the square** involves changing the first two terms of a quadratic relation of the form $y = ax^2 + bx + c$ into a perfect square while maintaining the balance of the original relation.

completing the square
- a process for expressing $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$

Example 1 Complete the Square

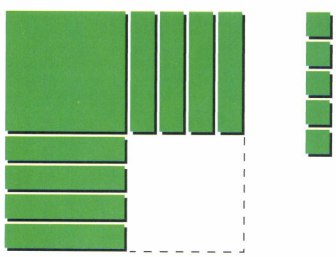
a) Rewrite $y = x^2 + 8x + 5$ in the form $y = a(x - h)^2 + k$.
b) Write the coordinates of the vertex of the parabola.
c) Sketch a graph of the relation. Label the vertex, the axis of symmetry, and two other points.

Solution

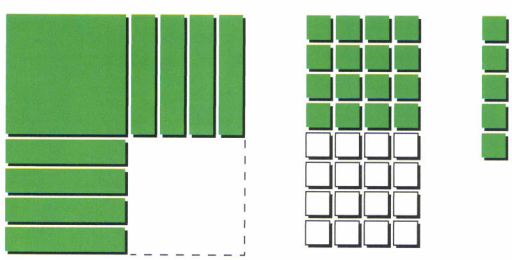
a) Method 1: Use Algebra Tiles

Create a perfect square using the first two terms in the quadratic expression $x^2 + 8x + 5$.

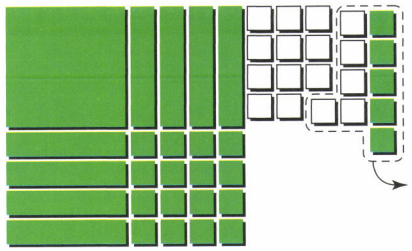
Arrange one x^2-tile and eight x-tiles so that the side lengths are equal. Place the five unit tiles to the side.



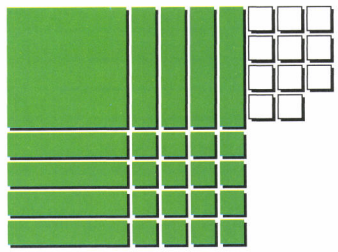
To complete the perfect square, you need to add 16 unit tiles.
In order to preserve the original quadratic expression, you must also add 16 negative unit tiles.



Complete the square and collect the unit tiles. Remove zero pairs.



$x^2 + 8x + 5 = (x + 4)^2 - 11$



The relation $y = x^2 + 8x + 5$ in the form $y = a(x - h)^2 + k$ is $y = (x + 4)^2 - 11$.

Method 2: Use Algebraic Symbols

Rewrite the relation in the form $y = a(x - h)^2 + k$ by completing the square.

$y = x^2 + 8x + 5$

$= (x^2 + 8x) + 5$ — Group the first two terms.

$= (x^2 + 8x + 4^2 - 4^2) + 5$ — To make a perfect square trinomial inside the brackets, add the square of half of 8, or 4^2. To balance the equation, also subtract 4^2.

$= (x^2 + 8x + 4^2) - 4^2 + 5$ — Take -4^2 outside the brackets.

$= (x + 4)^2 - 16 + 5$ — Factor the perfect square trinomial and simplify.

$= (x + 4)^2 - 11$

The relation $y = x^2 + 8x + 5$ in the form $y = a(x - h)^2 + k$ is $y = (x + 4)^2 - 11$.

b) The vertex is (h, k), or $(-4, -11)$.

c) The equation of the axis of symmetry is $x = -4$.

To find another point on the graph, let x take any value.

Let $x = 0$.

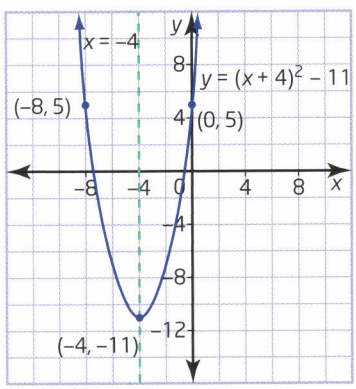
$y = 0^2 + 8(0) + 5$

$= 5$

Therefore, (0, 5) is a point on the parabola.

Due to symmetry, another point is the *partner* to this, (−8, 5).

Plot the three points and complete the sketch.



Looking at the x-coordinate of this point, I know that it is located 4 units to the right of the axis of symmetry. So, its partner is located 4 units to the left of the axis of symmetry and has the same y-coordinate.

Example 2 Find a Maximum or a Minimum

Find the maximum or minimum point of the parabola with equation $y = 2x^2 + 12x + 11$.

Solution

Method 1: Complete the Square

When the coefficient of the x^2-term is not 1, the first step is to factor the coefficient of x^2 from the first two terms. Then, complete the square within the brackets.

$y = 2x^2 + 12x + 11$

$= 2(x^2 + 6x) + 11$ — Factor 2 from the first two terms to make the coefficient of the x^2-term 1.

$= 2(x^2 + 6x + 3^2 - 3^2) + 11$ — To make a perfect square trinomial inside the brackets, add the square of half of 6, or 3^2. Subtract the same value to balance the equation.

$= 2(x^2 + 6x + 3^2) - 2(3^2) + 11$ — Take -3^2 outside of the brackets by multiplying by 2.

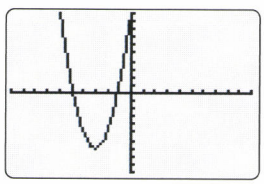
$= 2(x + 3)^2 - 18 + 11$ — Factor the perfect square trinomial and simplify.

$= 2(x + 3)^2 - 7$

The equation $y = 2(x + 3)^2 - 7$ is of the form $y = a(x - h)^2 + k$. The vertex is $(-3, -7)$. It is a minimum point, since a is positive.

Method 2: Use a Graphing Calculator

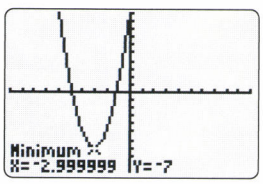
Enter the equation using [Y=].
Press [ZOOM] and select **6:ZStandard**.



You can see that the parabola has a minimum. This is because the coefficient of x^2 is positive.

Use the Minimum operation of a graphing calculator to find the coordinates of the vertex.

- Press [2nd] [CALC] to display the **CALCULATE** menu, and select **3:minimum**.
- Move the cursor to the left of the vertex and press [ENTER].
- Move the cursor to the right of the vertex and press [ENTER].
- Move the cursor close to the vertex and press [ENTER].

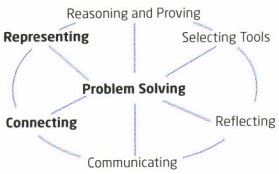


The calculator will give you the approximate ordered pair that best represents the minimum point of the graph.

The minimum point is $(-3, -7)$.

Technology Tip

The TI-83 Plus or TI-84 Plus graphing calculator displays cursor coordinates as eight-character numbers, which may include a negative sign. When **Float** is selected in the **MODE** settings, X and Y are displayed with a maximum accuracy of eight digits. Since the Minimum and Maximum operations are only calculated to an accuracy of 1×10^{-5} or 0.000 01, the result displayed on the graphing calculator screen may not be accurate to all eight displayed digits.



Example 3 Path of a Ball

The path of a ball is modelled by the equation $y = -x^2 + 2x + 3$, where x is the horizontal distance, in metres, from a fence and y is the height, in metres, above the ground.

a) What is the maximum height of the ball, and at what horizontal distance does it occur?

b) Sketch a graph to represent the path of the ball.

Solution

a) $y = -x^2 + 2x + 3$

$= -1(x^2 - 2x) + 3$ — Factor –1 from the first two terms.

$= -1(x^2 - 2x + (-1)^2 - (-1)^2) + 3$ — To complete the square inside the brackets, add the square of half of –2, or $(-1)^2$. Subtract the same value to balance the equation.

$= -1(x^2 - 2x + (-1)^2) - (-1)(-1)^2 + 3$ — Take $-(-1)^2$ outside the brackets by multiplying by –1.

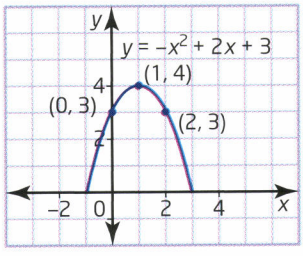
$= -1(x - 1)^2 + 1 + 3$ — Factor the perfect square trinomial and simplify.

$= -(x - 1)^2 + 4$

The equation $y = -(x - 1)^2 + 4$ is of the form $y = a(x - h)^2 + k$. The vertex is (1, 4). It is a maximum point since a is negative.

The maximum height of the ball is 4 m after it has been thrown a horizontal distance of 1 m.

b) The vertex is (1, 4).
When $x = 0$, $y = 3$.
By symmetry, the partner point to (0, 3) is (2, 3).



I can use these three points to graph the path of the ball. The parabola will not go below the x-axis because the height of the ball is always positive.

Example 4 Maximize Revenue

Alex runs a snowboard rental business that charges \$12 per snowboard and averages 36 rentals per day. She discovers that for each \$0.50 decrease in price, her business rents out two additional snowboards per day. At what price can Alex maximize her revenue?

Solution

Let R represent the total revenue, in dollars.
Let x represent the number of \$0.50 decreases in price.

Then, the price, in dollars, can be calculated as $12 - 0.5x$ and the number of rentals can be calculated as $36 + 2x$.

Revenue is the product of the price and the number rented.
$R = (12 - 0.5x)(36 + 2x)$

To find the maximum revenue, expand the quadratic relation, and then complete the square.

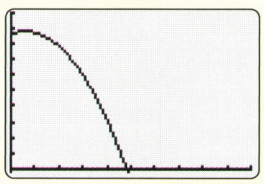
$$\begin{aligned} R &= (12 - 0.5x)(36 + 2x) \\ &= 432 + 6x - x^2 \\ &= -x^2 + 6x + 432 \\ &= -1(x^2 - 6x) + 432 \\ &= -1(x^2 - 6x + (-3)^2 - (-3)^2) + 432 \\ &= -1(x^2 - 6x + (-3)^2) - (-1)(-3)^2 + 432 \\ &= -1(x - 3)^2 + 9 + 432 \\ &= -(x - 3)^2 + 441 \end{aligned}$$

The relation reaches a maximum value of 441 when $x = 3$.

There should be three price reductions of \$0.50 to maximize the revenue.
$12 - 0.5(3) = 10.50$

A price of \$10.50 maximizes Alex's revenue.

I can use technology to graph both forms of the relation, $R = (12 - 0.5x)(36 + 2x)$ and $R = -(x - 3)^2 + 441$, and verify that they are equivalent.



Key Concepts

- You can rewrite a quadratic relation of the form $y = ax^2 + bx + c$ in the form $y = a(x - h)^2 + k$ by completing the square.
- For a quadratic relation in the form $y = a(x - h)^2 + k$, the vertex, (h, k), represents the maximum or minimum point of the parabola. The vertex is a minimum point when $a > 0$ and a maximum point when $a < 0$.
- Completing the square allows you to find the maximum or minimum point of a quadratic relation of the form $y = ax^2 + bx + c$ algebraically.
- You can use a graphing calculator to find the maximum or minimum point by using the Maximum or Minimum operation on the graph of the quadratic relation.

Communicate Your Understanding

C1 Describe the steps needed to complete the square for each relation.

a) $y = x^2 + 10x + 15$

b) $y = -2x^2 - 4x - 5$

C2 Identify the vertex of each relation in question C1. How do you know whether it is a maximum or a minimum point?

C3 Explain how to graph an equation of the form $y = a(x - h)^2 + k$.

Practise

For help with questions 1 to 6, see Example 1.

1. Use algebra tiles to rewrite each relation in the form $y = a(x - h)^2 + k$ by completing the square.

a) $y = x^2 + 2x + 5$

b) $y = x^2 + 4x + 7$

c) $y = x^2 + 6x + 3$

2. Determine the value of c that makes each expression a perfect square.

a) $x^2 + 6x + c$ **b)** $x^2 + 14x + c$

c) $x^2 - 12x + c$ **d)** $x^2 - 10x + c$

e) $x^2 + 2x + c$ **f)** $x^2 - 80x + c$

3. Rewrite each relation in the form $y = a(x - h)^2 + k$ by completing the square.

a) $y = x^2 + 6x - 1$

b) $y = x^2 + 2x + 7$

c) $y = x^2 + 10x + 20$

d) $y = x^2 + 2x - 1$

e) $y = x^2 - 6x - 4$

f) $y = x^2 - 8x - 2$

g) $y = x^2 - 12x + 8$

4. Find the vertex of each quadratic relation by completing the square.

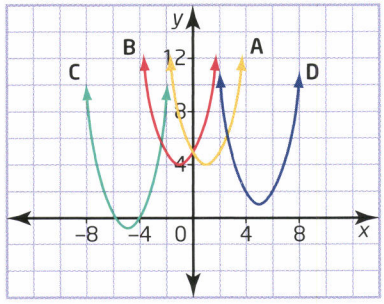
a) $y = x^2 + 6x + 2$

b) $y = x^2 + 12x + 30$

c) $y = x^2 - 8x + 13$

d) $y = x^2 - 6x + 17$

5. Match each graph with the appropriate equation.



a) $y = (x - 5)^2 + 1$

b) $y = (x - 1)^2 + 4$

c) $y = (x + 1)^2 + 4$

d) $y = (x + 5)^2 - 1$

6. Find the vertex of each parabola. Sketch the graph, labelling the vertex, the axis of symmetry, and two other points.

a) $y = x^2 + 10x + 20$

b) $y = x^2 - 16x + 60$

For help with questions 7 to 9, see Example 2.

7. Rewrite each relation in the form $y = a(x - h)^2 + k$ by completing the square.

a) $y = -x^2 + 80x - 100$

b) $y = -x^2 - 6x + 4$

c) $y = 3x^2 + 90x + 50$

d) $y = 2x^2 - 16x + 15$

e) $y = -7x^2 + 14x - 3$

8. Find the maximum or minimum point of each parabola by completing the square.

a) $y = -x^2 - 10x - 9$

b) $y = -x^2 + 14x - 50$

c) $y = 2x^2 + 120x + 75$

d) $y = 3x^2 - 24x + 10$

e) $y = -5x^2 - 200x - 120$

9. **Use Technology** Use a graphing calculator to find the maximum or minimum point of each parabola, rounded to the nearest tenth.

a) $y = x^2 + 6x - 1$

b) $y = 0.2x^2 - 1.5x + 6.3$

c) $y = -1.6x^2 + 4.3x - 5.2$

d) $y = \frac{1}{2}x^2 - \frac{1}{8}x + \frac{1}{2}$

e) $y = -57x^2 + 91x - 13$

f) $y = 144x^2 + 25x + 14$

For help with questions 10 and 11, see Example 3.

10. Find the vertex of each parabola. Sketch the graph, and label the vertex and two other points.

a) $y = -x^2 - 2x - 6$

b) $y = 4x^2 + 24x + 41$

c) $y = 5x^2 - 30x + 41$

d) $y = -3x^2 + 12x - 13$

e) $y = 2x^2 + 8x + 3$

11. Find the vertex of each parabola. Sketch the graph, and label the vertex and two other points.

a) $y = -2x^2 - 3x + 7$

b) $y = 3x^2 - 9x + 11$

c) $y = -x^2 + 8x - 10$

d) $y = 4x^2 - 16x + 11$

e) $y = -5x^2 - 30x - 48$

Connect and Apply

12. The path of a ball is modelled by the equation $y = -x^2 + 4x + 1$, where x is the horizontal distance, in metres, travelled and y is the height, in metres, of the ball above the ground. What is the maximum height of the ball, and at what horizontal distance does it occur?

13. **Use Technology** A football is kicked at an angle of 30° to the ground, at an initial speed of 20 m/s, from a height of 1 m. Two quadratic relations can be used to model the height, in metres, above the ground:

With respect to time, t, in seconds, the height is given by $h = -4.9t^2 + 10t + 1$.

With respect to the horizontal distance, x, in metres, the height is given by $h = -0.0163x^2 + 0.5774x + 1$.

Use a graphing calculator to verify that the maximum height is the same with both models.

At what time and horizontal distance does the maximum height occur?

Did You Know?

Galileo was a mathematics professor at the University of Pisa, in Italy. At the beginning of the 17th century, he discovered the connection between quadratic equations and acceleration.

14. A diver dives from the 3-m board at a swimming pool. Her height, y, in metres, above the water in terms of her horizontal distance, x, in metres, from the end of the board is given by $y = -x^2 + 2x + 3$. What is the diver's maximum height?

15. The cost, in dollars, of operating a machine per day is given by the formula $C = 2t^2 - 84t + 1025$, where t is the time, in hours, the machine operates. What is the minimum cost of running the machine? For how many hours must the machine run to reach this minimum cost?

For help with question 16, see Example 4.

16. An artisan can sell 120 garden ornaments per week at \$4 per ornament. For each \$0.50 decrease in price, he can sell 20 more ornaments.

a) Determine algebraic expressions for the price of a garden ornament and the number of ornaments sold.

b) Write an equation for the revenue using your expressions from part a).

c) Use your equation from part b) to find what price the artisan should charge to maximize revenue.

d) **Use Technology** Graph both forms of the relation from parts b) and c) to verify that they are equivalent.

17. Find the maximum or minimum point of each parabola by completing the square.

a) $y = 1.5x^2 + 6x - 7$

b) $y = -0.1x^2 - 2x + 1$

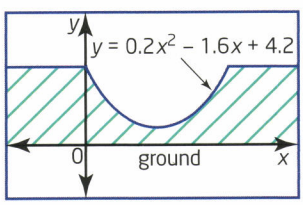
c) $y = 0.3x^2 + 3x$

d) $y = -1.25x^2 + 5x$

e) $y = 0.5x^2 - 6x + 12$

f) $y = -0.02x^2 - 0.6x - 9$

18. **Chapter Problem** A manufacturer decides to build a half-pipe with a parabolic cross section modelled by the relation $y = 0.2x^2 - 1.6x + 4.2$, where x is the horizontal distance, in metres, from the platform, and y is the height, in metres, above the ground.



Complete the square to find the depth of the half-pipe.

19. Find the two missing values (b, c, and/or h) in each equation.

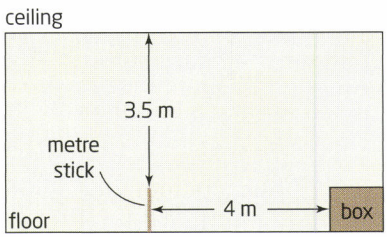
a) $x^2 + 8x + c = (x + h)^2$

b) $x^2 + bx + 36 = (x + h)^2$

c) $x^2 + bx + c = (x - 5)^2 + 2$

20. The drag on a small aircraft is made up of induced drag from the wings as they produce lift and parasitic drag from the airframe. Over a limited speed range, the drag, d, in newtons, produced by a speed, v, in kilometres per hour, can be modelled by the quadratic relation $d = 0.15v^2 - 9v + 195$. Determine the speed that results in minimum drag.

21. Fred fires a toy spring from the top of a metre stick toward a cardboard box 4 m away and 1 m tall. The spring follows a path modelled by the relation $y = -x^2 + 10x - 21$. The ceiling of the room is 3.5 m above the point at which the spring is launched. Can the spring hit the box without hitting the ceiling first?



22. **Use Technology** Studies show that employees on an assembly line become more efficient as their level of training goes up. In one company, the number of products, P, produced per day, at a level of training of t hours, follows the quadratic model $P = -0.375t^2 + 10.25t - 8$, for $0 \leq t \leq 18$. Use a graphing calculator to determine what level of training will give the maximum productivity, rounded to the nearest tenth.

23. A pipe cleaner is 20 cm long. It is bent into a rectangle. Use a quadratic model to determine the dimensions that give the maximum area.

24. A field is bounded on one side by a river. The field is to be enclosed on three sides by a fence, to create a rectangular enclosure. The total length of fence to be used is 200 m. Use a quadratic model to determine the dimensions of the enclosure of maximum area.

Extend

25. Use Technology A projectile is propelled upward at various angles from the ground (known as angles of elevation), with initial velocity of 10 m/s. Use graphing technology to compare the maximum heights of the projectiles with the angles of elevation shown in the table.

Angle of Elevation	Equation
20°	$h = -4.9t^2 + 3.42t$
30°	$h = -4.9t^2 + 5.00t$
40°	$h = -4.9t^2 + 6.43t$
50°	$h = -4.9t^2 + 7.66t$
60°	$h = -4.9t^2 + 8.66t$
70°	$h = -4.9t^2 + 9.40t$

26. A parabola has a y-intercept of 5 and contains the points $(2, -3)$ and $(-1, 12)$. What are the coordinates of the vertex?

27. Verify that the x-coordinate of the vertex of a parabola of the form $y = ax^2 + bx + c$ is $-\frac{b}{2a}$.

28. Math Contest The maximum area for an equilateral triangle inscribed in a circle of radius R is $\frac{3\sqrt{3}}{4}R^2$. Determine the side length of the triangle.

29. Math Contest Two names from four friends, Jane, Farhad, Sonia, and Mehta, are drawn from a hat for two tickets to a concert. What is the probability that Jane and Farhad go to the concert together?

A $\frac{2}{3}$ **B** $\frac{1}{2}$

C $\frac{1}{4}$ **D** $\frac{1}{6}$

E $\frac{1}{12}$

30. Math Contest In 1920, the young nephew of American mathematician Edward Kasner coined the word *googol* to mean a really big number.
1 googol $= 10^{100}$
1000^{1000} is equivalent to

A 1000 googols **B** 3000 googols

C 30 googols **D** googol^{30}

E googol$^{\text{googol}}$

Did You Know?

The founders of the Internet search engine Google intended to use googol as their name. However, investors misspelled the name on a cheque.

6.2 Solve Quadratic Equations

The flight path of a toy rocket, a ball, or any projectile can be predicted using a quadratic model. This model can also be used to determine when and where a projectile will land. For example, the path of a stone thrown into a ravine is modelled by the quadratic relation $y = -x^2 + 5x + 84$, where x is the distance, in metres, it travels horizontally and y is the height, in metres, above the river at the bottom of the ravine. You can solve for x using a variety of methods, including factoring a **quadratic equation**.

quadratic equation

- an equation in the form $ax^2 + bx + c = 0$, where a, b, and c are real numbers and $a \neq 0$

Investigate

How can factoring help you solve a quadratic equation?

1. If the product of two numbers is zero, what must be true about one or both of the numbers?
2. If $a \times b = 0$, what must be true about the value of a or the value of b or both?
3. How do steps 1 and 2 relate to solving the equation $(x - 3)(x + 5) = 0$?
4. **a)** Solve the equation $(x - 3)(x + 5) = 0$.
   b) Explain why your solutions cause the left side of the equation to equal zero.
5. Use your method to solve each equation.
   a) $(x + 2)(x + 9) = 0$ **b)** $(2x + 5)(3x - 4) = 0$
6. **a)** How is the equation $x^2 + 6x + 8 = 0$ different from those in steps 3 to 5? Describe the extra steps needed to solve the equation.
   b) Solve the equation.
7. **Reflect** Describe how you can use factoring to solve a quadratic equation.

Example 1 Solve by Factoring

Solve for x. Check your answers by substitution.

a) $x^2 + 9x + 14 = 0$

b) $2x^2 + 5x = 0$

c) $6x^2 - x = 15$

Solution

a) $x^2 + 9x + 14 = 0$

$(x + 7)(x + 2) = 0$ — Factor the left side.

$x + 7 = 0$ or $x + 2 = 0$ — One factor or the other must equal zero.

$x = -7$ or $x = -2$

Check.

For $x = -7$:

$\text{L.S.} = x^2 + 9x + 14$ $\qquad$ $\text{R.S.} = 0$

$= (-7)^2 + 9(-7) + 14$

$= 49 - 63 + 14$

$= 0$

$\text{L.S.} = \text{R.S.}$

For $x = -2$:

$\text{L.S.} = x^2 + 9x + 14$ $\qquad$ $\text{R.S.} = 0$

$= (-2)^2 + 9(-2) + 14$

$= 4 - 18 + 14$

$= 0$

$\text{L.S.} = \text{R.S.}$

The solutions, or **roots**, are -7 and -2.

b) $2x^2 + 5x = 0$

$x(2x + 5) = 0$ — Remove the common factor.

$x = 0$ or $2x + 5 = 0$ — One factor or the other must equal zero.

$2x = -5$

$x = -\frac{5}{2}$

The roots are 0 and $-\frac{5}{2}$. The check of these roots is left to the reader.

c) $6x^2 - x = 15$

$6x^2 - x - 15 = 0$ — Write in the form $ax^2 + bx + c = 0$.

$6x^2 - 10x + 9x - 15 = 0$ — To factor the left side, break up the middle term and factor by grouping.

$2x(3x - 5) + 3(3x - 5) = 0$

$(3x - 5)(2x + 3) = 0$

$3x - 5 = 0$ or $2x + 3 = 0$ — One factor or the other must equal zero.

$3x = 5$ or $2x = -3$

$x = \frac{5}{3}$ or $x = -\frac{3}{2}$

Literacy Connections

To solve an equation means to find the values of the variable that make the statement true. This is also called finding the roots of the equation.

root (of an equation)

- the value of the variable that makes an equation true
- the same as the solution of an equation

To factor $6x^2 - x - 15$, I must find two integers whose product is $6 \times (-15)$, or -90, and whose sum is -1. The integers -10 and 9 work.

Check.

For $x = \frac{5}{3}$:

L.S. $= 6x^2 - x$ **R.S.** $= 15$

$= 6\left(\frac{5}{3}\right)^2 - \frac{5}{3}$

$= 6\left(\frac{25}{9}\right) - \frac{5}{3}$

$= \frac{50}{3} - \frac{5}{3}$

$= 15$

L.S. $=$ **R.S.**

For $x = -\frac{3}{2}$:

L.S. $= 6x^2 - x$ **R.S.** $= 15$

$= 6\left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)$

$= 6\left(\frac{9}{4}\right) + \frac{3}{2}$

$= \frac{27}{2} + \frac{3}{2}$

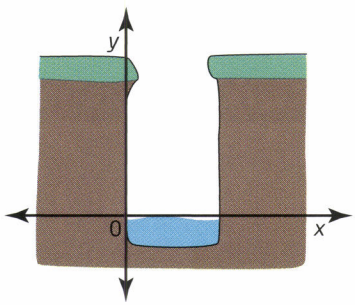
$= 15$

L.S. $=$ **R.S.**

The roots are $\frac{5}{3}$ and $-\frac{3}{2}$.

Example 2 Path of a Stone

The path of a stone thrown into a ravine is modelled by the quadratic relation $y = -x^2 + 5x + 84$, where x represents the distance, in metres, travelled horizontally and y represents the height, in metres, above the surface of the river at the bottom of the ravine. How far does the stone travel horizontally before it hits the water?



Solution

When the stone hits the water, its height is 0 m. So, let $y = 0$.

$-x^2 + 5x + 84 = 0$

$\frac{-x^2}{-1} + \frac{5x}{-1} + \frac{84}{-1} = \frac{0}{-1}$ Divide both sides of the equation by -1.

$x^2 - 5x - 84 = 0$

$(x + 7)(x - 12) = 0$ Factor the left side.

$x + 7 = 0$ or $x - 12 = 0$ One factor or the other must equal zero.

$x = -7$ or $x = 12$

Since x represents a distance, it cannot be negative. So, reject the root -7.

Check $x = 12$.

L.S. $= -x^2 + 5x + 84$ **R.S.** $= 0$

$= -(12)^2 + 5(12) + 84$

$= -144 + 60 + 84$

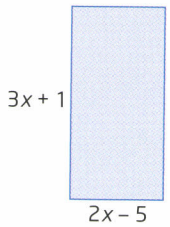
$= 0$

L.S. $=$ **R.S.**

The stone travelled 12 m horizontally before it hit the water.

Example 3 Dimensions of a Rectangle

A rectangle has dimensions $3x + 1$ and $2x - 5$.
Its area is 1150 cm^2. What are its dimensions?



Solution

Method 1: Use Pencil and Paper

Substitute the area and expressions for the dimensions into the formula $A = l \times w$.

$$1150 = (3x + 1)(2x - 5)$$

$$1150 = 6x^2 - 13x - 5$$ Expand and simplify the right side.

$$6x^2 - 13x - 1155 = 0$$ Write in the form $ax^2 + bx + c = 0$.

$$6x^2 - 90x + 77x - 1155 = 0$$ Factor the left side.

$$6x(x - 15) + 77(x - 15) = 0$$

$$(x - 15)(6x + 77) = 0$$

$x - 15 = 0$ or $6x + 77 = 0$ One factor or the other must equal zero.

$x = 15$ or $x = -\frac{77}{6}$

Since the dimensions cannot be negative, reject the solution $-\frac{77}{6}$.

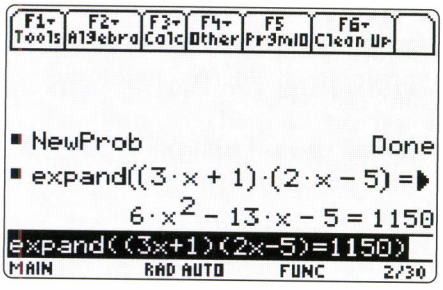
Check $x = 15$.
Find the dimensions of the rectangle.

$3x + 1$ $\quad$ $2x - 5$
$= 3(15) + 1$ $\quad$ $= 2(15) - 5$
$= 46$ $\quad$ $= 25$

Check that these dimensions give an area of 1150 cm^2.
$46 \times 25 = 1150$
Therefore, the dimensions of the rectangle are 46 cm by 25 cm.

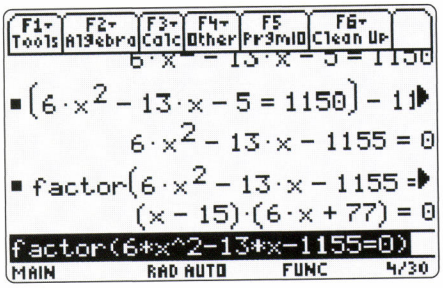
Method 2: Use a Computer Algebra System (CAS)

In the Home screen, use the **Expand** function with the area equation $(3x + 1)(2x - 5) = 1150$.



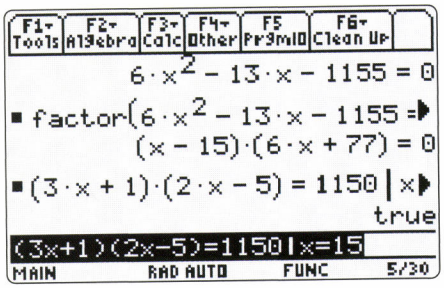
Express the equation in the form $ax^2 + bx + c = 0$.
Use the up cursor key to highlight the expanded form. Press ◆ ↑ for [COPY]. Cursor down to the command line. Press ◆ ESC for [PASTE]. Put brackets around the equation and subtract 1150. Press ENTER.

COPY and PASTE the new form of the equation into the command line. Cursor to the beginning of the equation. Press F2. Select **2:factor(**. At the end of the command line, press) ENTER.


F1 Tools
F2 Algebra
F3 Calc
F4 Other
F5 PrgmIO
F6 Clean Up
$6 \cdot x^2 - 13 \cdot x - 5 = 1150$
$(6 \cdot x^2 - 13 \cdot x - 5 = 1150) - 11$
$6 \cdot x^2 - 13 \cdot x - 1155 = 0$
$\text{factor}(6 \cdot x^2 - 13 \cdot x - 1155 =$
$(x - 15) \cdot (6 \cdot x + 77) = 0$
factor(6*x^2-13*x-1155=0)
MAIN RAD AUTO FUNC 4/30

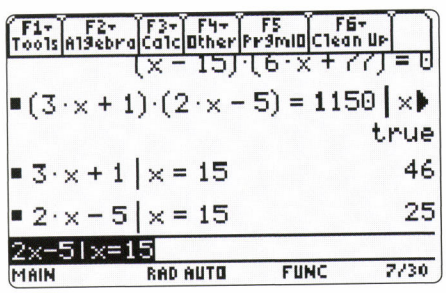
Note the two factors. The second factor results in a negative solution, and is rejected. Therefore, $x = 15$ is the solution.

To check the solution, type the original equation, followed by the *such that* symbol, | , and then $x = 15$. Press ENTER.


F1 Tools
F2 Algebra
F3 Calc
F4 Other
F5 PrgmIO
F6 Clean Up
$6 \cdot x^2 - 13 \cdot x - 1155 = 0$
$\text{factor}(6 \cdot x^2 - 13 \cdot x - 1155 =$
$(x - 15) \cdot (6 \cdot x + 77) = 0$
$(3 \cdot x + 1) \cdot (2 \cdot x - 5) = 1150 \mid x$
true
(3x+1)(2x-5)=1150|x=15
MAIN RAD AUTO FUNC 5/30

Note that the CAS returns a value of *true*.

Next, calculate the dimensions of the rectangle.
Type $3x + 1 \mid x = 15$ and press ENTER.
Type $2x - 5 \mid x = 15$ and press ENTER.


F1 Tools
F2 Algebra
F3 Calc
F4 Other
F5 PrgmIO
F6 Clean Up
$(x - 15) \cdot (6 \cdot x + 77) = 0$
$(3 \cdot x + 1) \cdot (2 \cdot x - 5) = 1150 \mid x$
true
$3 \cdot x + 1 \mid x = 15$ 46
$2 \cdot x - 5 \mid x = 15$ 25
2x-5|x=15
MAIN RAD AUTO FUNC 7/30

Check that these dimensions give an area of 1150 cm^2.
$46 \times 25 = 1150$
Therefore, the dimensions of the rectangle are 46 cm by 25 cm.

Key Concepts

- If two factors have a product of zero, then one or both of the factors must equal zero.
- A quadratic equation has degree two and has a single unknown. For example, $x^2 + 2x - 3 = 0$ is a quadratic equation.
- Some quadratic equations can be solved by factoring.
- To solve a quadratic equation by factoring, first write the equation in the form $ax^2 + bx + c = 0$, and then factor the left side. Next, set each factor equal to zero, and solve for the unknown.

  For example,

$$\begin{aligned} x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0 \\ (x + 3)(x - 1) &= 0 \\ x + 3 = 0 \quad &\text{or} \quad x - 1 = 0 \\ x = -3 \quad &\text{or} \quad x = 1 \end{aligned}$$

- The solutions to a quadratic equation are also known as the roots of the equation.

Communicate Your Understanding

C1 When you are factoring to solve a quadratic equation such as $x^2 + 2x - 3 = 0$, the right side needs to be equal to zero. Why?

C2 Describe how you would solve each quadratic equation by factoring.

a) $3x^2 + 12x + 9 = 0$

b) $2x^2 - 11x = -15$

Practise

For help with questions 1 to 5, see Example 1.

1. Solve.

a) $(x + 5)(x + 2) = 0$

b) $(x - 3)(x + 4) = 0$

c) $(x - 1)(x - 7) = 0$

d) $x(x + 9) = 0$

e) $(2x + 3)(x - 5) = 0$

f) $(2x - 1)(3x + 4) = 0$

g) $(3x - 5)(4x - 3) = 0$

2. Solve and check.

a) $x^2 + 8x + 12 = 0$

b) $h^2 + 9h + 18 = 0$

c) $m^2 + 3m = 0$

d) $w^2 - 18w + 56 = 0$

e) $x^2 - 2x = 0$

f) $c^2 - 17c + 30 = 0$

g) $n^2 + 9n - 22 = 0$

h) $y^2 - 11y = 0$

3. Solve.

a) $3x^2 + 28x + 9 = 0$

b) $4k^2 + 19k + 15 = 0$

c) $8y^2 - 22y + 15 = 0$

d) $16b^2 - 1 = 0$

e) $10m^2 + 30m = 0$

f) $4x^2 - 12x + 9 = 0$

4. Solve.

a) $x^2 + 5x = -4$

b) $8c + 15 = -c^2$

c) $k^2 = 13k - 12$

d) $b^2 + 1 = -2b$

e) $m^2 = 300 - 20m$

f) $y^2 = 7y$

5. Solve.

a) $2m^2 = -7m - 6$

b) $9x^2 = x + 8$

c) $4y^2 - 12y = -9$

d) $-5 = 2p - 16p^2$

e) $12m^2 = 10 - 37m$

f) $3w^2 + 22w = -7$

For help with questions 6 and 7, see Example 2.

6. Solve.

a) $-x^2 - 10x - 16 = 0$

b) $3t^2 + 24t + 45 = 0$

c) $6d^2 + 15d = -9$

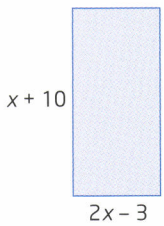
d) $-10g^2 + 32g = 6$

Connect and Apply

7. A basketball is tossed from the top of a 3-m wall. The path of the basketball is defined by the relation $y = -x^2 + 2x + 3$, where x represents the horizontal distance travelled, in metres, and y represents the height, in metres, above the ground. How far has the basketball travelled horizontally when it lands on the ground?

For help with question 8, see Example 3.

8. A rectangle has dimensions $x + 10$ and $2x - 3$. Determine the value of x that gives an area of 54 cm^2.



9. Write a quadratic equation in factored form for each situation.

a) The roots of the equation are 5 and 4.

b) The roots of the equation are -2 and 3.

10. a) Write a quadratic equation in the form $ax^2 + bx + c = 0$ with roots of 6 and -7.

b) What would happen to the roots if you multiplied both sides of the equation in part a) by 3? Explain.

11. Write a quadratic equation with roots of $\frac{2}{3}$ and $-\frac{4}{5}$ in the form $ax^2 + bx + c = 0$, where a, b, and c are integers.

12. a) Create an example of a quadratic equation that can be factored and solved with integer solutions.

b) Create an example of a quadratic equation that can be factored and solved with non-integer solutions.

13. Create an example of a quadratic equation that cannot be solved by factoring. Explain why it cannot be factored.

14. The hypotenuse of a right triangle measures 29 cm. One leg is 1 cm shorter than the other. What are the lengths of the legs?

15. For the equation $3n^2 = 15n$, Chris suggested dividing both sides by $3n$, leaving $n = 5$. Are there any other values that satisfy this equation? What is wrong with Chris's method? What is a more appropriate method?

16. The sum of the first n even natural numbers can be found using the formula $S = n(n + 1)$.

a) Verify the formula for $n = 1$ and $n = 2$.

b) What is the sum of the first five even natural numbers?

c) The sum of the first n even natural numbers is 306. What is the value of n?

17. At the Mini Market, as the price of milk drops, sales increase. On an average day, a 4-L bag of milk costs \$3.90, and the store sells an average of 120 bags. For each \$0.10 reduction in price of a 4-L bag, sales increase by 20 bags per day. The price and value of sales can be modelled as follows, where n is the number of \$0.10 price reductions.

Price, in dollars: $3.90 - 0.10n$
Number of bags: $120 + 20n$

The total revenue is the product of the price and the number of bags sold. Find how many price reductions will result in revenue of \$700.

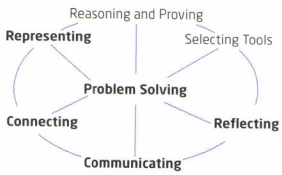
Did You Know?

According to the Canadian Dairy Information Centre, in 2005 the per capita consumption of fluid milk in Canada was 84.00 L. Fluid milk includes 3.25%, 2%, 1%, skim, chocolate, and buttermilk.

For the same year, Finland had the greatest per capita retail milk consumption, 116.99 L, while China had the least, 2.55 L.

Achievement Check

18. The length of the base of a rectangular prism is 2 m greater than the width, and the height of the prism is 15 m.



a) Write an algebraic expression for the volume of the rectangular prism. Express the relation in the form $y = a(x - h)^2 + k$.

b) If the volume of the prism is 2145 m^3, write an equation to model the situation.

c) Solve the equation by factoring. What are the dimensions of the base of the rectangular prism?

d) Make up a similar problem. Trade with a partner and solve each other's problem.

Extend

19. The length of a rectangular plot of land is 7 m greater than its width. The diagonal is 8 m greater than the width of the plot of land. What are the dimensions of the plot of land?

20. Ralph is opening a BMX bike repair shop. His accountant models his profit, P, with the equation $P = 1125(t - 1)^2 - 4500$, where t is the number of years of operation. During the first 5 years of operation, when is Ralph's shop predicted to make a loss? a profit?

21. a) Solve the equation $y^2 + 3y + 2 = 0$. How is the equation $y^2 + 3yx + 2x^2 = 0$ related?

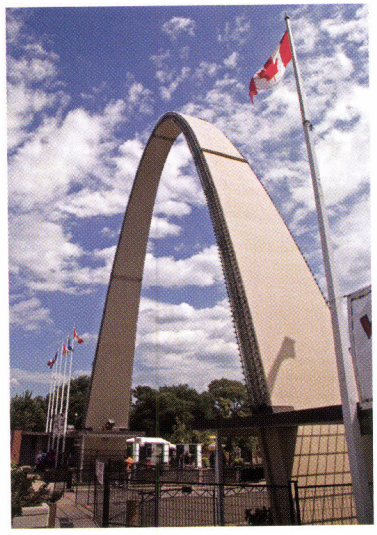
b) Solve for y in terms of x.

i) $y^2 + 3yx + 2x^2 = 0$

ii) $5y^2 - 6yx - 8x^2 = 0$

iii) $\frac{1}{9}y^2 - \frac{1}{3}yx + \frac{1}{4}x^2 = 0$

6.3 Graph Quadratics Using the *x*-Intercepts



In Chapter 4, you graphed quadratic relations of the form $y = a(x - r)(x - s)$ by using the *x*-intercepts. In this section, you will use *x*-intercepts to graph parabolas with equations of the form $y = ax^2 + bx + c$. With these skills, you will be able to calculate, for example, the width of an arch or the time it takes for a projectile to land on the ground.

Investigate A

Tools
- grid paper

How can you use factoring to graph a quadratic relation?

1. Compare the information you can get from quadratic relations of the form $y = ax^2 + bx + c$, $y = (ax + b)(cx + d)$, and $y = a(x - h)^2 + k$. Which form is easiest to use for finding the vertex? the *x*-intercepts? the *y*-intercept?

2. Consider the quadratic relation $y = (x - 2)(x + 4)$.
   - **a)** Explain how you would find the *x*-intercepts. Find and plot the *x*-intercepts.
   - **b)** How can you find the *x*-coordinate of the vertex? Explain why this method is appropriate.
   - **c)** Find the *x*-coordinate of the vertex. Then, calculate the *y*-coordinate.
   - **d)** Plot the vertex and sketch the graph of the quadratic relation.

3. For each relation, find the *x*-intercepts, the equation of the axis of symmetry, and the coordinates of the vertex. Use the information to sketch the graph.
   - **a)** $y = (x + 5)(x + 1)$ **b)** $y = 2(2x - 1)(2x + 3)$

4. How would you find the *x*-intercepts and graph the following quadratic relations?
   - **a)** $y = x^2 + 3x - 10$ **b)** $y = 6x^2 - 17x + 5$

5. **Reflect** Summarize your findings about how to use factoring to graph a quadratic relation of the form $y = ax^2 + bx + c$.

Investigate B

How are the *x*-intercepts of a quadratic relation related to the roots of the corresponding quadratic equation?

Tools

- TI-83 Plus or TI-84 Plus graphing calculator

1. a) Graph the relation $y = x^2 - 3x - 4$ as **Y1**. Press ZOOM and select **6:ZStandard.**

b) Find the **zeros** of the relation.

- Press 2nd [CALC] to display the **CALCULATE** menu, and select **2:zero**.
- Move the cursor to the left of one *x*-intercept and press ENTER.
- Move the cursor to the right of the *x*-intercept and press ENTER.
- Move the cursor closer to the *x*-intercept and press ENTER.

The calculator will give you the approximate ordered pair that best represents the location of the *x*-intercept. Repeat for the other *x*-intercept.

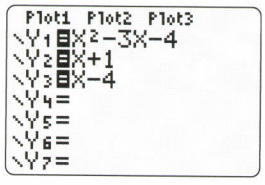
c) What is the *y*-coordinate of each zero?

2. For the quadratic relation $y = x^2 - 3x - 4$, the corresponding quadratic equation is $x^2 - 3x - 4 = 0$.

a) For which points on the graph of $y = x^2 - 3x - 4$ are the *x*-coordinates roots of $x^2 - 3x - 4 = 0$? Explain.

b) How can you use these points to write the factors of the corresponding quadratic equation?

c) Graph the factors as **Y2** and **Y3**. Explain why the graphs of the lines pass through the *x*-intercepts of the graph of the quadratic relation.



d) Graph the product of the factors as **Y4**. Does a fourth graph appear in the window? What conclusion can you make?

3. Use the graphs to connect the values of factors to the quadratic relation.

a) Find the values of the factors at $x = 1$. Verify that their product lies on the quadratic.

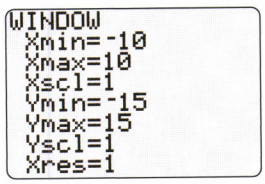
- Press 2nd [CALC], select **1:value**, type 1, and press ENTER.
- Use the up and down cursor keys to move from one graph to another.

b) Use the procedure in part a) to examine the values of the factors at the *x*-intercepts, $x = -1$ and $x = 4$. Do both factors have a value of 0 at the *x*-intercepts? Explain.

4. Will a constant factor affect the graph of the quadratic relation?

   a) Graph $y = 2(x + 1)(x - 4)$ as **Y5**. Use the window settings shown.

   b) How is this graph different from that of $y = (x + 1)(x - 4)$? How is it similar?



5. **Reflect** Explain how the x-intercepts, or zeros, of a quadratic relation are related to the corresponding quadratic equation.

Many quadratic relations of the form $y = ax^2 + bx + c$ can be factored to find the x-intercepts. By the definition of the x-intercepts, $y = 0$. For any product to equal zero, one or both of the factors must equal zero. For this reason, the x-intercepts are also known as the zeros of the quadratic relation.

For example, for the relation $y = (x - 2)(x + 5)$, the zeros occur when $y = 0$. If $(x - 2)(x + 5) = 0$, then either $x - 2 = 0$ or $x + 5 = 0$.

You can solve for x. Either $x = 2$ or $x = -5$.

Example 1 Use Factoring to Graph a Quadratic Relation

Find the x-intercepts and the coordinates of the vertex. Then, sketch each parabola.

a) $y = x^2 + 8x + 7$

b) $y = 2x^2 - x - 6$

c) $y = -x^2 + 6x - 9$

Solution

a) To find the x-intercepts, factor and solve the corresponding quadratic equation by letting $y = 0$.

$y = x^2 + 8x + 7$
$0 = x^2 + 8x + 7$
$0 = (x + 1)(x + 7)$
$x + 1 = 0$ or $x + 7 = 0$
$x = -1$ or $x = -7$

The x-intercepts are -1 and -7.

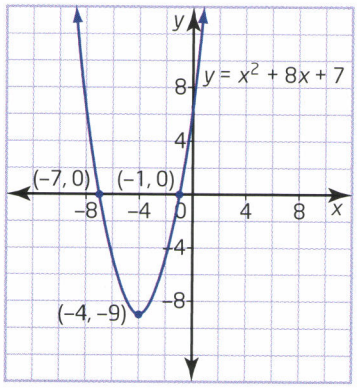
Due to the symmetric property of a parabola, the x-coordinate of the vertex is on the axis of symmetry that passes through the midpoint of the line segment connecting the x-intercepts.

$$x = \frac{-1 + (-7)}{2}$$
$$= -4$$

To find the y-coordinate of the vertex, substitute $x = -4$ into the equation of the parabola.

$$\begin{aligned} y &= x^2 + 8x + 7 \\ &= (-4)^2 + 8(-4) + 7 \\ &= 16 - 32 + 7 \\ &= -9 \end{aligned}$$

The coordinates of the vertex are $(-4, -9)$.



b) Factor and solve the corresponding quadratic equation.

$$\begin{aligned} y &= 2x^2 - x - 6 \\ 0 &= 2x^2 - x - 6 \\ 0 &= 2x^2 + 3x - 4x - 6 \\ 0 &= x(2x + 3) - 2(2x + 3) \\ 0 &= (2x + 3)(x - 2) \end{aligned}$$

$2x + 3 = 0$ or $x - 2 = 0$

$x = -\frac{3}{2}$ or $x = 2$

To factor $2x^2 - x - 6$, I need to find two integers whose product is $2 \times (-6)$, or -12, and whose sum is -1. These integers are 3 and -4.

The x-intercepts are $-\frac{3}{2}$ and 2.

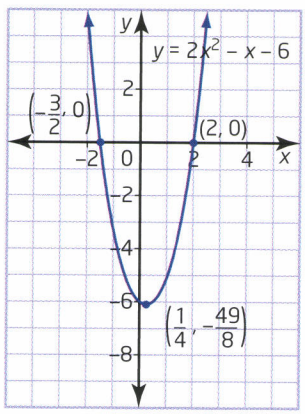
Now, find the x-coordinate of the vertex.

$$\begin{aligned} x &= \frac{-\frac{3}{2} + 2}{2} \\ &= \frac{\frac{1}{2}}{2} \\ &= \frac{1}{4} \end{aligned}$$

Then, use its value to find the y-coordinate.

$$\begin{aligned} y &= 2x^2 - x - 6 \\ &= 2\left(\frac{1}{4}\right)^2 - \frac{1}{4} - 6 \\ &= \frac{2}{16} - \frac{1}{4} - 6 \\ &= \frac{1}{8} - \frac{2}{8} - \frac{48}{8} \\ &= -\frac{49}{8} \end{aligned}$$

The coordinates of the vertex are $\left(\frac{1}{4}, -\frac{49}{8}\right)$.



c) Factor and solve the corresponding quadratic equation.

$y = -x^2 + 6x - 9$
$0 = -x^2 + 6x - 9$
$0 = -(x^2 - 6x + 9)$
$0 = -(x - 3)^2$

After I factor out −1, the expression inside the brackets is a perfect square trinomial.

There is a single x-intercept at $x = 3$. Because of symmetry, all points except the vertex have a partner. Since this point has no partner, it is the vertex. Therefore, the coordinates of the vertex are $(3, 0)$.

Find two more points on the parabola.

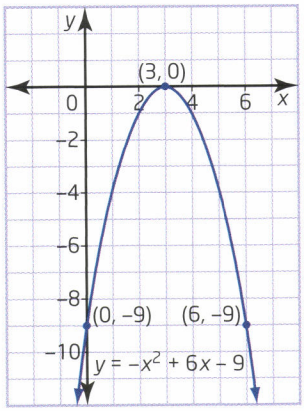
Let $x = 0$. Then $y = -9$.

I can choose any value for x, but $x = 0$ is easy to calculate.

The partner point to $(0, -9)$ is $(6, -9)$.

I know that the axis of symmetry is a vertical line passing through the vertex. So, by comparing the x-coordinates of $(0, -9)$ and the vertex $(3, 0)$, I know that $(0, -9)$ is located 3 units to the left of the axis of symmetry. Then, its partner is located 3 units to the right of the axis of symmetry and has the same y-coordinate.

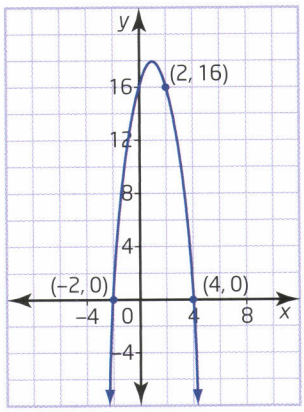
Plot the three points and complete the sketch.



In Example 1, you looked at parabolas with two x-intercepts or one x-intercept. You will investigate parabolas with no x-intercepts in Section 6.4.

Example 2 Find an Equation From a Graph

Write an equation in the form $y = ax^2 + bx + c$ for the parabola shown.



Solution

Since the x-intercepts are -2 and 4, the equation has the form $y = a(x + 2)(x - 4)$.

To find a, substitute $(x, y) = (2, 16)$.

$16 = a(2 + 2)(2 - 4)$
$16 = -8a$
$-2 = a$
$y = -2(x + 2)(x - 4)$
$= -2(x^2 - 2x - 8)$
$= -2x^2 + 4x + 16$

Example 3 Design an Arch

To commemorate the 100th anniversary of the Newtonville Fair, an entrance arch will be built. The design engineer uses the equation $h = -d^2 + 16$ to model the arch, where h is the height, in metres, above the ground and d is the horizontal distance, in metres, from the centre of the arch.

a) How wide and how tall is the arch?

b) For what values of d is the relation valid? Explain.

c) If a width of 2.5 m is needed per line-up at the entrance, how many line-ups can there be?

Solution

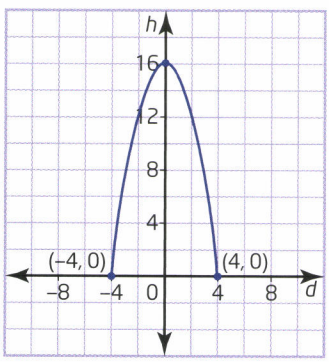
a) $h = -d^2 + 16$
$0 = -(d^2 - 16)$
$0 = -(d - 4)(d + 4)$

The expression inside the brackets is a difference of squares.

The d-intercepts are 4 and -4, which are symmetric about the h-axis.

The arch is 8 m wide at its base.

At $d = 0$, $h = 16$. The arch is 16 m tall at the centre.



b) The relation is valid for $-4 \leq d \leq 4$. Negative values of d represent distances on the opposite side of the axis of symmetry of those on the positive side.

c) $8 \div 2.5 = 3.2$
The entrance arch could accommodate three line-ups.

Key Concepts

- The x-intercepts of some quadratic relations in the form $y = ax^2 + bx + c$ can be found by factoring.
- A quadratic relation can be graphed by using its x-intercepts.

  For example, to find the x-intercepts of the quadratic relation $y = x^2 - 2x - 3$, factor and solve the corresponding quadratic equation by letting $y = 0$.

  $x^2 - 2x - 3 = 0$
  $(x - 3)(x + 1) = 0$
  $x - 3 = 0$ or $x + 1 = 0$
  $x = 3$ or $x = -1$

  Since the roots of $x^2 - 2x - 3 = 0$ are 3 and -1, the x-intercepts of $y = x^2 - 2x - 3$ are 3 and -1.
- The x-intercepts, or zeros, of a quadratic relation $y = ax^2 + bx + c$ are the solutions to $ax^2 + bx + c = 0$.
- The x-coordinate of the vertex is on the axis of symmetry, which passes through the midpoint of the line segment connecting the x-intercepts. Substitute this value into the quadratic relation to find the y-coordinate of the vertex.
- When the x-intercepts and one other point are known, the information can be substituted into $y = a(x - r)(x - s)$ to find an equation representing the relation.

Communicate Your Understanding

C1 Describe the steps involved in finding the x-intercepts of the parabola with equation $y = x^2 - 3x - 70$.

C2 Explain why you can find the x-coordinate of the vertex of a quadratic relation by using the midpoint of the line segment connecting the x-intercepts.

Practise

For help with questions 1 to 5, see Example 1.

1. Find the x-intercepts.

a) $y = x^2 + 5x + 6$
b) $y = x^2 - 11x + 28$
c) $y = x^2 + 9x$
d) $y = x^2 + 5x - 24$
e) $y = x^2 - 2x - 8$
f) $y = x^2 + 9x - 36$

2. Find the x-intercepts.

a) $y = 4x^2 + 20x + 9$
b) $y = 8x^2 - 6x$
c) $y = 6x^2 - 11x - 7$
d) $y = 5x^2 + 6x - 8$
e) $y = 3x^2 - 13x + 4$
f) $y = 4x^2 - 20x + 25$

3. Find the x-intercepts and the vertex of each parabola. Then, sketch its graph.

a) $y = x^2 + 9x + 14$

b) $y = x^2 - 6x + 8$

c) $y = -x^2 - 4x + 5$

d) $y = -x^2 - 5x$

4. Find the x-intercepts and the vertex of each parabola. Then, sketch its graph.

a) $y = x^2 - 9$

b) $y = -x^2 + 10x - 9$

c) $y = x^2 - 12x + 36$

d) $y = 16 - x^2$

5. Find the zeros and the vertex of each parabola. Then, sketch its graph. Check your results with a graphing calculator.

a) $y = 2x^2 + 15x + 7$

b) $y = 12x^2 - 16x + 5$

c) $y = -8x^2 - 13x + 6$

d) $y = 8x^2 + 17x + 9$

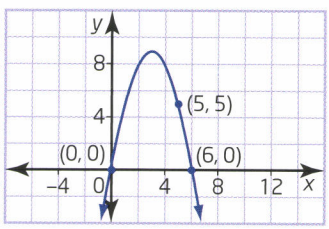
e) $y = 16x^2 - 24x + 9$

f) $y = -4x^2 - 18x - 8$

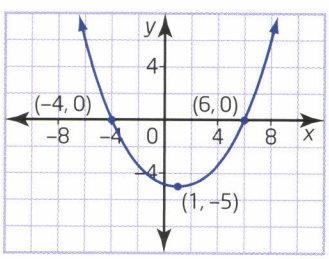
For help with questions 6 and 7, see Example 2.

6. Write an equation, in the form $y = ax^2 + bx + c$, to represent each parabola.

a)

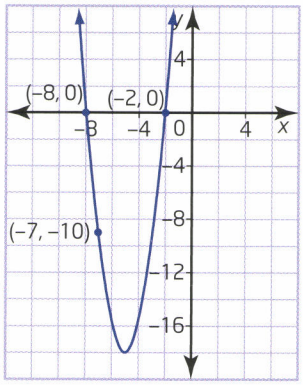


b)

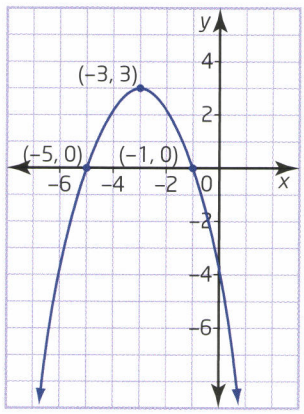


7. Write an equation, in the form $y = ax^2 + bx + c$, to represent each parabola.

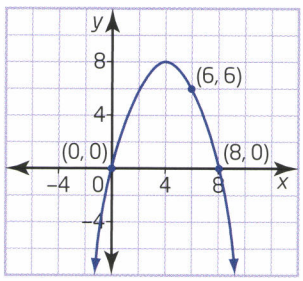
a)



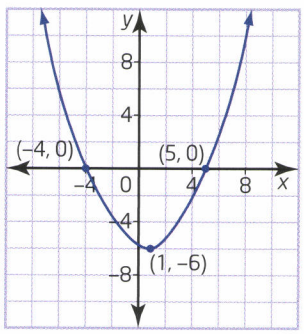
b)



c)



d)



Connect and Apply

For help with question 8, see Example 3.

8. The parabolic cross section of a metal garage is modelled by the relation $h = -d^2 + 4$, where h is the height, in metres, above the ground and d is the horizontal distance, in metres, from the centre of the garage.

a) How wide and how tall is the garage?

b) Sketch a graph to represent the cross section.

c) For what values of d is the relation valid? Explain.

9. The path of a toy rocket is defined by the relation $y = -3x^2 + 11x + 4$, where x is the horizontal distance, in metres, travelled and y is the height, in metres, above the ground.

a) Determine the zeros of the relation.

b) For what values of x is the relation valid?

c) How far has the rocket travelled horizontally when it lands on the ground?

d) What is the maximum height of the rocket above the ground, to the nearest hundredth of a metre?

10. A parabolic arch supports a bridge. The arch is 200 m wide at its base and 4 m tall in the middle.

a) Show this information on a graph by placing the vertex on the y-axis so that the x-axis represents the base of the arch.

b) How high is a point on the arch that is 20 m horizontally from one end?

11. A parabola has vertex $(-3, 7)$, and one x-intercept is -11. Find the other x-intercept and the y-intercept.

12. The path of a kicked ball can be defined by the equation $h = -\frac{1}{35}d^2 + \frac{16}{35}d + \frac{36}{35}$, where h is its height, in metres, above the ground and d is the horizontal distance, in metres.

a) Find the d-intercepts.

b) Sketch a graph of this relation.

c) For what values of d is this relation valid? Explain.

d) At what height was the ball kicked?

e) How far had the ball travelled when it landed on the ground?

13. The paved surface of a road has a parabolic cross section, so that rainwater flows off the surface to the edges. The cross section of the road can be modelled as $d = -\frac{1}{125}w^2 + \frac{2}{25}w$, where d is the depth, in metres, of the road relative to the gravel base, and w is the width, in metres, from one curb.

a) Sketch a graph of this relation.

b) For what values of w is this relation valid? Explain.

c) How wide is the road?

d) How high does the road rise from the gravel base?

14. Explain what must be true for a parabola to have only one x-intercept.

15. Find the value(s) of n so that each quadratic relation has only one zero. Then, sketch its graph.

a) $y = x^2 + nx + 25$

b) $y = -x^2 + nx - 9$

c) $y = 16x^2 - 8x + n$

16. In a science competition, two teams of students each design a machine to catapult a marble horizontally as far as possible, from a height of 3 m. The following equations represent the paths of the marbles for the teams, where x is the horizontal distance, in metres, and y is the height, in metres, above the ground.

The Marble Heads: $y = -x^2 + 2x + 3$
The XY Team: $y = -2x^2 + x + 3$

a) Which team's marble travels farther? How much farther?

b) At what horizontal distance are both marbles at the same height?

c) Which team's marble flies higher?

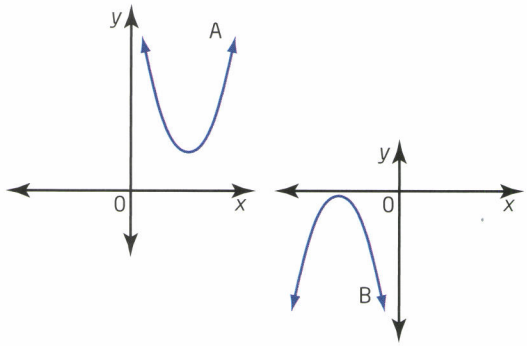
Extend

17. The cross section of the roof of an airplane hangar is parabolic. Its profile can be defined by the equation $y = -\frac{1}{100}x^2 + 50$, where y is the height, in metres, above the ground and x is the width, in metres, from the centre of the building. A Bombardier Canadair CRJ-700 jet airplane has a wingspan of 23.24 m.

a) How many CRJ-700s can fit side by side inside the hangar?

b) If you assume that the wings are 2 m above the floor of the hangar, does your answer in part a) change?

18. a) The parabolas shown do not have any x-intercepts. Use your knowledge of transformations to create an example of an equation that could be represented by each parabola.



b) Explain how you know that the equations for these parabolas cannot be factored.

19. A parabola has equation $y = 4x^2 + x + c$. One x-intercept is -3. What is the value of c?

20. Sketch the graph of $y = x^2 + 4$. Are there any x-intercepts? Use this information to explain why $a^2 + b^2$ cannot be factored.

21. Math Contest Solve for x.

a) $3^{2x} - 12(3^x) + 27 = 0$

b) $2^{2x} - 3(2^{x+3}) + 128 = 0$

22. Math Contest

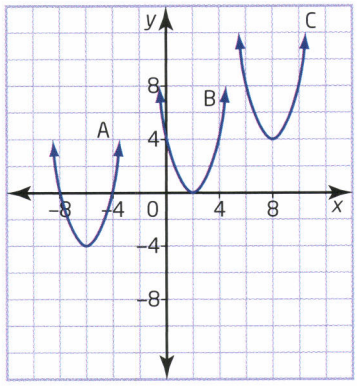
a) In a family of three children, determine the probability that there are exactly two boys.

b) John is one of three children. What is the probability that there are exactly two boys in John's family?

c) Explain the similarities or differences in your answers to parts a) and b).

6.4 The Quadratic Formula

Quadratic equations that can be factored are fairly simple to solve. But what about quadratics that cannot be factored? The Greek mathematicians Euclid (300 BCE) and Pythagoras (500 BCE) both derived geometric solutions to a quadratic equation. A general solution for quadratic equations, using numbers, was derived in about 700 AD by the Hindu mathematician Brahmagupta. The general formula used today was derived in about 1100 AD by another Hindu mathematician, Bhaskara. He was also the first to recognize that any positive number has two square roots, one positive and one negative.



For each parabola shown, how many real roots does the related quadratic equation have?

Investigate A

Tools
- grid paper

How can you use the process of completing the square to solve quadratic equations?

1. Graph the relation $y = x^2 - 9$. How many x-intercepts are there?

2. **a)** Solve the equation $x^2 - 9 = 0$ by isolating x^2 first.
   b) How many roots are there?
   c) How do the roots relate to the graph in step 1?

3. **a)** Graph the relation $y = (x + 2)^2 - 9$. How many x-intercepts are there?
   b) Solve the equation $(x + 2)^2 - 9 = 0$ by isolating the perfect square, $(x + 2)^2$, then taking the square root of both sides, and finally solving for x. Remember that there are two square roots.
   c) How many roots are there? How do the roots relate to the graph in part a)?

4. **a)** Solve the equation $x^2 + 10x + 16 = 0$ by first completing the square.
   b) Describe the steps you used to solve for x.

5. **Reflect** Would the method you used to solve the quadratic equation in step 4 work for any quadratic equation? Explain.

6. Use your method to solve each quadratic equation.
   a) $3x^2 + 30x + 48 = 0$
   b) $5x^2 - 20x - 52 = 0$

Investigate B

How can you use completing the square to derive a formula for solving quadratic equations?

By completing the square, you can develop a formula that can be used to solve any quadratic equation. Copy the calculations in the table. Describe the steps in the example and how they relate to the development of the **quadratic formula**.

Example	Quadratic Formula
$2x^2 + 5x + 1 = 0$	$ax^2 + bx + c = 0$
$x^2 + \frac{5}{2}x + \frac{1}{2} = 0$	$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$
$\left(x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2 - \left(\frac{5}{4}\right)^2\right) + \frac{1}{2} = 0$	$\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2\right) + \frac{c}{a} = 0$
$\left(x^2 + \frac{5}{2}x + \left(\frac{5}{4}\right)^2\right) - \left(\frac{5}{4}\right)^2 + \frac{1}{2} = 0$	$\left(x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right) - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$
$\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{1}{2} = 0$	$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0$
$\left(x + \frac{5}{4}\right)^2 - \frac{25}{16} + \frac{8}{16} = 0$	$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{4ac}{4a^2} = 0$
$\left(x + \frac{5}{4}\right)^2 - \frac{17}{16} = 0$	$\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2} = 0$
$\left(x + \frac{5}{4}\right)^2 = \frac{17}{16}$	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
$x + \frac{5}{4} = \pm\sqrt{\frac{17}{16}}$	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$
$x + \frac{5}{4} = \pm\frac{\sqrt{17}}{4}$	$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$
$x = -\frac{5}{4} \pm \frac{\sqrt{17}}{4}$	$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-5 \pm \sqrt{17}}{4}$	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

quadratic formula

- a formula for determining the roots of a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$
- $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 1 Real Roots

Use the quadratic formula to solve each quadratic equation. Where necessary, round to the nearest hundredth.
Verify graphically using technology.

a) $2x^2 + 9x + 6 = 0$

b) $4x^2 - 12x = -9$

Solution

a) For $2x^2 + 9x + 6 = 0$, $a = 2$, $b = 9$, and $c = 6$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-9 \pm \sqrt{9^2 - 4(2)(6)}}{2(2)}$$

$$= \frac{-9 \pm \sqrt{81 - 48}}{4}$$

$$= \frac{-9 \pm \sqrt{33}}{4}$$

The exact roots are $\frac{-9 + \sqrt{33}}{4}$ and $\frac{-9 - \sqrt{33}}{4}$.

You can also express the answers as approximate roots.

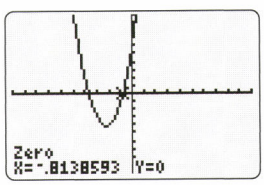
$$x = \frac{-9 + \sqrt{33}}{4} \quad \text{or} \quad x = \frac{-9 - \sqrt{33}}{4}$$

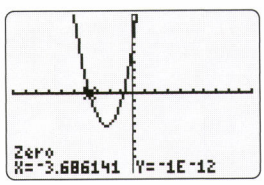
$$\doteq -0.81 \qquad \doteq -3.69$$

```
(-9+√(33))/4
      -.8138593384
(-9-√(33))/4
      -3.686140662
```

The approximate roots are -0.81 and -3.69, to the nearest hundredth.

Use the Zero operation of a graphing calculator to verify that the roots are the zeros of the related quadratic relation $y = 2x^2 + 9x + 6$.





Technology Tip

When a graphing calculator displays a result such as −1E−12, it represents a number in scientific notation.

−1E−12 = -1×10^{-12}

b) First, write $4x^2 - 12x = -9$ in the form $ax^2 + bx + c = 0$.

$4x^2 - 12x + 9 = 0$

For $4x^2 - 12x + 9 = 0$, $a = 4$, $b = -12$, and $c = 9$.

I can see that the trinomial on the left side of the equation is a perfect square, so I could have solved by factoring.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-12) \pm \sqrt{(-12)^2 - 4(4)(9)}}{2(4)}$$

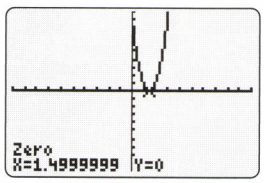
$$= \frac{12 \pm \sqrt{144 - 144}}{8}$$

$$= \frac{12 \pm \sqrt{0}}{8}$$

$$= \frac{12}{8}$$

$$= \frac{3}{2}$$

The root is $\frac{3}{2}$ or 1.5.



When the x-intercepts are known, you can find the x-coordinate of the vertex by finding the midpoint of the line segment connecting the x-intercepts.

Use the two x-intercepts from the quadratic formula.

$$x = \frac{\frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}}{2}$$

$$= \frac{\frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a}}{2}$$

$$= \frac{\frac{-2b}{2a}}{2}$$

$$= \frac{-2b}{4a}$$

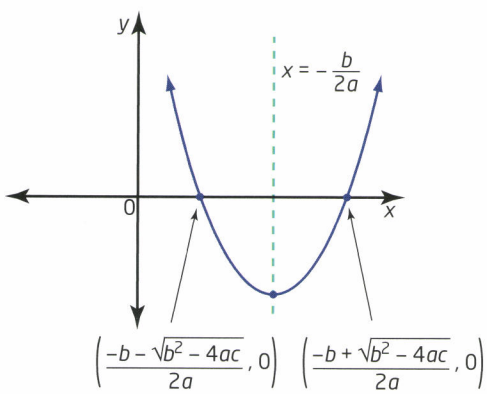
$$= -\frac{b}{2a}$$

The square root expressions have a zero sum.

$\sqrt{b^2 - 4ac} - \sqrt{b^2 - 4ac} = 0$

The x-coordinate of the vertex is $-\frac{b}{2a}$.

This also gives the equation of the axis of symmetry, $x = -\frac{b}{2a}$.



Example 2 Use the Quadratic Formula to Sketch a Parabola

Find the x-intercepts, the vertex, and the equation of the axis of symmetry of the quadratic relation $y = -5x^2 + 8x - 3$. Sketch the parabola.

Solution

To find the x-intercepts, let $y = 0$ and use the quadratic formula to solve the quadratic equation.

For $-5x^2 + 8x - 3 = 0$, $a = -5$, $b = 8$, and $c = -3$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-8 \pm \sqrt{8^2 - 4(-5)(-3)}}{2(-5)}$$

$$= \frac{-8 \pm \sqrt{64 - 60}}{-10}$$

$$= \frac{-8 \pm \sqrt{4}}{-10}$$

$$= \frac{-8 \pm 2}{-10}$$

Therefore,

$$x = \frac{-8 - 2}{-10} \quad \text{or} \quad x = \frac{-8 + 2}{-10}$$

$$= 1 \qquad\qquad = \frac{3}{5} \text{ or } 0.6$$

The x-intercepts are 0.6 and 1.

To find the x-coordinate of the vertex, use $x = -\frac{b}{2a}$.

$$x = -\frac{8}{2(-5)}$$

$$= -\frac{8}{-10}$$

$$= \frac{4}{5} \text{ or } 0.8$$

I can check this one mentally. The x-coordinate of the midpoint of the line segment connecting the x-intercepts, 0.6 and 1, is 0.8.

Substitute $x = 0.8$ into $y = -5x^2 + 8x - 3$ to find the y-coordinate of the vertex.

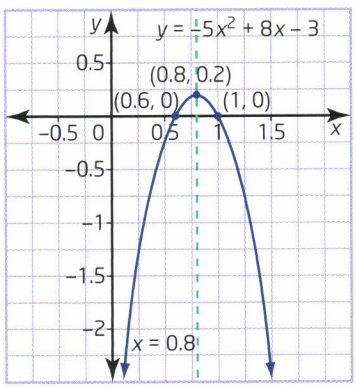
$y = -5(0.8)^2 + 8(0.8) - 3$

$= -3.2 + 6.4 - 3$

$= 0.2$

The coordinates of the vertex are (0.8, 0.2).

The axis of symmetry has equation $x = 0.8$.



Example 3 Connect a Parabola and No Real Roots

A parabola has equation $y = (x - 2)^2 + 3$.

a) State the coordinates of the vertex, the equation of the axis of symmetry, and the direction of opening.

b) Determine the x-intercepts. Verify using the quadratic formula.

c) Sketch the parabola.

Solution

a) The vertex is (2, 3). The equation of the axis of symmetry is $x = 2$. The parabola opens upward, since a is positive.

b) Since the vertex of the parabola is above the x-axis and it opens upward, it has no x-intercepts. This can be verified using the quadratic formula after expanding and simplifying the original equation.

$$\begin{aligned} y &= (x - 2)^2 + 3 \\ &= x^2 - 4x + 4 + 3 \\ &= x^2 - 4x + 7 \end{aligned}$$

Let $y = 0$ and use the quadratic formula with $a = 1$, $b = -4$, and $c = 7$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(7)}}{2(1)}$$

$$= \frac{4 \pm \sqrt{16 - 28}}{2}$$

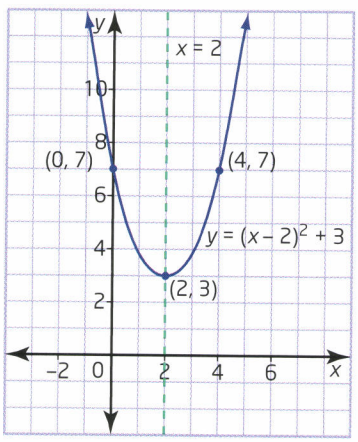
$$= \frac{4 \pm \sqrt{-12}}{2}$$

Since the square root of a negative number is not a real number, there are no real roots.
Therefore, the parabola has no x-intercepts.

c) Let $x = 0$. A second point on the curve is (0, 7).

Looking at the x-coordinate of this point, I know that it is located 2 units to the left of the axis of symmetry. So, its partner is located 2 units to the right of the axis of symmetry and has the same y-coordinate.

Then, due to symmetry, the partner point on the parabola is (4, 7).



Example 4 Path of a Basketball

The path of a basketball after it is thrown from a height of 1.5 m above the ground is given by the equation $h = -0.25d^2 + 2d + 1.5$, where h is the height, in metres, and d is the horizontal distance, in metres.

a) How far has the ball travelled horizontally, to the nearest tenth of a metre, when it lands on the ground?

b) Find the horizontal distance when the basketball is at a height of 4.5 m above the ground.

Solution

a) When the basketball lands on the ground, the height is 0 m.
Let $h = 0$.
For $-0.25d^2 + 2d + 1.5 = 0$, $a = -0.25$, $b = 2$, and $c = 1.5$.

$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

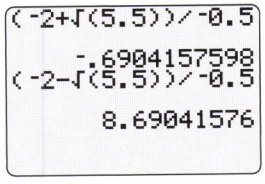
$$= \frac{-2 \pm \sqrt{2^2 - 4(-0.25)(1.5)}}{2(-0.25)}$$

$$= \frac{-2 \pm \sqrt{4 + 1.5}}{-0.5}$$

$$= \frac{-2 \pm \sqrt{5.5}}{-0.5}$$

So, $d \doteq -0.7$ or $d \doteq 8.7$.

Since d represents distance, it must be positive.
The basketball has travelled a horizontal distance of about 8.7 m when it lands on the ground.



b) Let $h = 4.5$.

$-0.25d^2 + 2d + 1.5 = 4.5$

$-0.25d^2 + 2d - 3 = 0$

I need to express the equation in the form $ax^2 + bx + c = 0$.

For $-0.25d^2 + 2d - 3 = 0$, $a = -0.25$, $b = 2$, and $c = -3$.

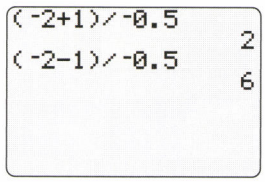
$$d = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4(-0.25)(-3)}}{2(-0.25)}$$

$$= \frac{-2 \pm \sqrt{4 - 3}}{-0.5}$$

$$= \frac{-2 \pm \sqrt{1}}{-0.5}$$

$$= \frac{-2 \pm 1}{-0.5}$$



So, $d = 2$ or $d = 6$.

The basketball will be at a height of 4.5 m twice along its parabolic path: on the way up at a horizontal distance of 2 m and on the way down at a horizontal distance of 6 m.

Key Concepts

- A quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$, can be solved for x using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- The x-coordinate of the vertex of a parabola is $-\frac{b}{2a}$, and the equation of the axis of symmetry is $x = -\frac{b}{2a}$.

Solving Quadratic Equations: $ax^2 + bx + c = 0$

Method	When It Can Be Used	Comments
graphing	always	The solutions will not always be exact: this is best used only when an approximate answer is needed.
factoring	sometimes	Use when $c = 0$ or when factors are easily found.
completing the square	always	This is best used for equations of the form $x^2 + bx + c = 0$, where b is an even number.
quadratic formula	always	This method always gives exact solutions, but in some cases the other methods are easier to use.
computer algebra system (CAS)	always	Use the solve (or factor) function of the CAS.

Communicate Your Understanding

C1 Jan calculated the x-intercepts of a quadratic relation as $x = \frac{-2 \pm \sqrt{5}}{8}$.

a) What are the individual x-intercepts?

b) Describe how to use the x-intercepts to find the equation of the axis of symmetry.

C2 After using the quadratic formula, explain how you would know if a quadratic equation has

a) two real roots

b) one real root

c) no real roots

Practise

For help with questions 1 and 2, see Example 1.

1. Use the quadratic formula to solve each equation. Express answers as exact roots.

a) $7x^2 + 24x + 9 = 0$

b) $2x^2 + 4x - 7 = 0$

c) $4x^2 - 12x + 9 = 0$

d) $2x^2 - 7x = -4$

e) $3x^2 + 5x = 1$

f) $16x^2 + 24x = -9$

2. **Use Technology** Use the quadratic formula to solve. Express your answers as exact roots and as approximate roots, rounded to the nearest hundredth. Verify graphically with technology.

a) $3x^2 + 14x + 5 = 0$

b) $8x^2 + 12x + 1 = 0$

c) $4x^2 - 7x - 1 = 0$

d) $10x^2 - 45x - 7 = 0$

e) $-5x^2 + 16x - 2 = 0$

f) $-6x^2 + 17x + 5 = 0$

For help with questions 3 to 5, see Examples 2 and 3.

3. Find the x-intercepts, the vertex, and the equation of the axis of symmetry of each quadratic relation. Then, sketch the parabola.

a) $y = 5x^2 - 14x - 3$

b) $y = 2x^2 - 5x - 12$

c) $y = x^2 + 10x + 25$

d) $y = 9x^2 - 24x + 16$

e) $y = x^2 - 2x + 3$

f) $y = -x^2 - 3x - 3$

4. For each quadratic relation, state the coordinates of the vertex and the direction of opening. Then, determine how many x-intercepts the relation has.

a) $y = (x - 3)^2 + 2$

b) $y = -(x - 2)^2 + 4$

c) $y = 2(x + 4)^2 - 5$

d) $y = -2(x + 3)^2 - 1$

e) $y = (x - 5)^2$

5. Verify the number of x-intercepts for each relation in question 4 by using the quadratic formula to find the x-intercepts.

Connect and Apply

For help with question 6, see Example 4.

6. The path of a soccer ball after it is kicked from a height of 0.5 m above the ground is given by the equation $h = -0.1d^2 + d + 0.5$, where h is the height, in metres, above the ground and d is the horizontal distance, in metres.

a) How far has the soccer ball travelled horizontally, to the nearest tenth of a metre, when it lands on the ground?

b) Find the horizontal distance when the soccer ball is at a height of 2.6 m above the ground.

7. A ball is thrown upward at an initial velocity of 8.4 m/s, from a height of 1.5 m above the ground. The height of the ball, in metres, above the ground, after t seconds, is modelled by the equation $h = -4.9t^2 + 8.4t + 1.5$.

a) After how many seconds does the ball land on the ground? Round your answer to the nearest tenth of a second.

b) What is the maximum height, to the nearest metre, that the ball reaches?

8. Write an equation of a parabola, in the form $y = a(x - h)^2 + k$, satisfying each description. Then, write each relation in the form $y = ax^2 + bx + c$. Use graphing technology or the quadratic formula to verify that your equation satisfies the description.

a) two x-intercepts

b) one x-intercept

c) no x-intercept

9. Solve. Round answers to the nearest hundredth, where necessary.

a) $7x^2 - 12x = 9$

b) $4x^2 = 12 - 13x$

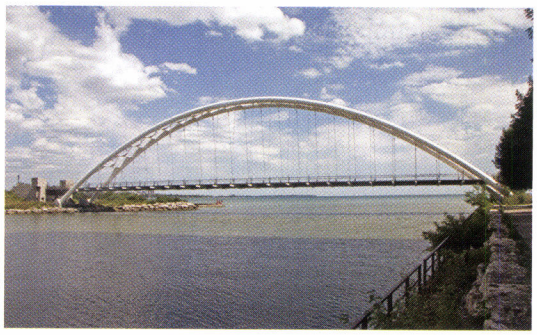
c) $4x^2 = 2.8x + 4.8$

d) $x(3x - 8) = -1$

e) $(x - 3)^2 = -2(x + 3)$

f) $(x + 3)^2 = (2x + 5)(2x - 5)$

10. The shape of the Humber River pedestrian bridge in Toronto can be modelled by the equation $y = -0.0044x^2 + 21.3$. All measurements are in metres. Determine the length of the bridge and the maximum height above the ground, to the nearest tenth of a metre.



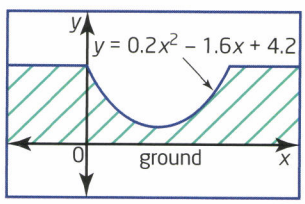
11. A toy rocket is launched from a 3-m platform, at 8.1 m/s. The height of the rocket is modelled by the equation $h = -4.9t^2 + 8.1t + 3$, where h is the height, in metres, above the ground and t is the time, in seconds.

a) After how many seconds will the rocket rise to a height of 6 m above the ground? Round your answer to the nearest hundredth.

b) When does the rocket fall again to a height of 6 m above the ground?

c) Use your answers from parts a) and b) to determine when the rocket reached its maximum height above the ground.

12. **Chapter Problem** The platforms on the ends of the half-pipe are at the same height.



a) How wide is the half-pipe?

b) How far would a skater have travelled horizontally after a drop of 2 m? Round to the nearest hundredth of a metre.

13. A shopping mall entrance contains a parabolic arch, modelled by the equation $h = -0.5(d - 8)^2 + 32$, where h is the height, in metres, above the floor and d is the distance, in metres, from one end of the arch. How wide is the arch at its base?

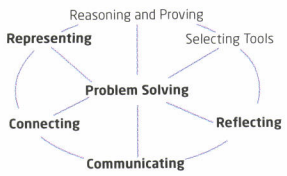
14. The fuel flowing to the engine of a small aircraft can be modelled by a quadratic equation over a limited range of speeds using the relation $f = 0.0048v^2 - 0.96v + 64$, where f represents the flow of fuel, in litres per hour, and v represents speed, in kilometres per hour.

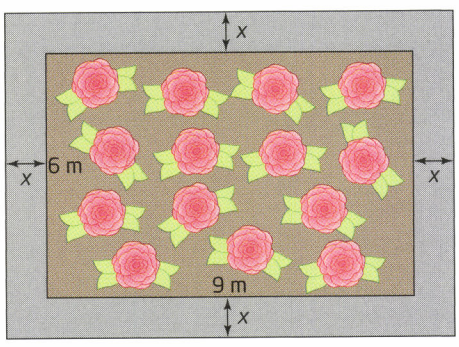
a) Show that this quadratic relation has no v-intercepts.

b) Determine the speed that minimizes fuel flow.

Achievement Check

15. The parks department is planning a new flower bed outside city hall. It will be rectangular with dimensions 9 m by 6 m. The flower bed will be surrounded by a path of constant width with the same area as the flower bed. Calculate the perimeter of the outside of the path.





a) Set up a quadratic equation to model the question.

b) Use the quadratic formula to solve the problem.

c) **Use Technology** Check your solution using a CAS or a graphing calculator.

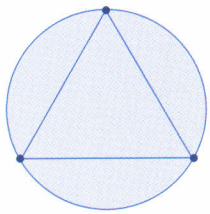
Extend

16. Find a quadratic equation with each pair of roots.

a) $x = \dfrac{14 \pm \sqrt{140}}{4}$

b) $x = \dfrac{-11 \pm \sqrt{145}}{6}$

17. Points are drawn on a circle.



a) If there are three points, how many line segments can be drawn joining any two points?

b) What if there are four points? five points? six points?

c) If there are n points, how many line segments can be drawn joining any two points?

d) How many points are needed in order to have at least 1000 line segments?

18. When $b^2 - 4ac = 0$, there is only one real root. Give an example showing why this is so.

19. **Math Contest** Find the points of intersection of $x^2 + y^2 = 16$ and $y = x^2 - 9$, and sketch a graph.

20. **Math Contest** What are the different possibilities for the number of points of intersection for a circle and a parabola? For each case, give an equation for the circle and for the parabola.

21. **Math Contest** Show that the roots of $x = 1 + \dfrac{1}{x}$ are negative reciprocals. The positive root of $x = 1 + \dfrac{1}{x}$ is called the *golden ratio*.

22. Math Contest A continued fraction is a fraction of the form $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cdots}}}$.

a) Evaluate the continued fraction $2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5}}}$.

b) Evaluate the infinite continued fraction $1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}$.

23. Math Contest In 1202, Leonardo of Pisa (1175–1250), better known as Fibonacci, published a book called *Liber Abaci*. In it, he showed a sequence of numbers now called the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, ...

a) Find the next three terms and a formation rule for this sequence.

b) Examine the ratios of successive terms of the Fibonacci sequence $\frac{(n+1)\text{th term}}{n\text{th term}}$. Conjecture a possible value for the ratio $\frac{1000\text{th term}}{999\text{th term}}$.

24. Math Contest Solve the system of equations.

$a - b + c = 0$
$a - c + d = 1$
$a + 2b + 2d = 0$
$b - d = 3$

Making Connections

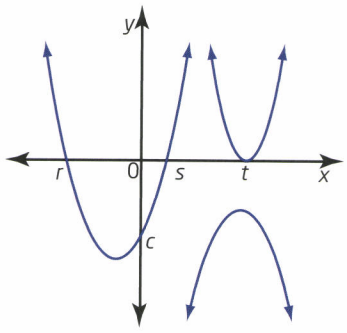
A mind map can provide a mental picture of how many of the concepts you have learned about quadratics in Chapters 4, 5, and 6 are connected. One is started here. Make a larger version for yourself. Complete the shaded parts and add any other details that help you.

Quadratic Relations

A $y = ax^2 + bx + c$
$a > 0$ means
$a < 0$ means
c is the

B $y = a(x - h)^2 + k$
a means the same as in form **A**
(h, k) is the

C $y = a(x - r)(x - s)$
a means the same as in form **A**
r and s are the



x-intercepts↔zeros↔roots

Quadratic Equations

$ax^2 + bx + c = 0$

If the quadratic expression factors:

- $a(x - r)(x - s) = 0$
  2 zeros are at
- $a(x - t)^2 = 0$ perfect square
  1 zero at t
- $a(x - p)(x + p) = 0$ difference of squares
  roots are
  axis of symmetry is

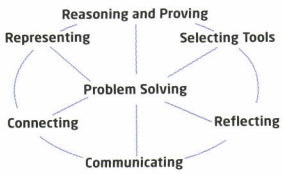
If no easy factors are found:

- complete the square to get form **B**
- use the quadratic formula
  if no real roots exist, then parabola

6.5 Solve Problems Using Quadratic Equations

In this chapter, you have learned a variety of methods for solving quadratic equations: graphing, factoring, completing the square, and the quadratic formula. You have applied these skills to solve problems related to situations that can be modelled by quadratic relations, such as paths of projectiles, shapes of parabolic structures, measurement problems involving area, and maximizing revenue.

Learning to use the most appropriate method in a given situation is an important step in becoming a good problem solver.



Example 1 Model the Path of a Toy Rocket

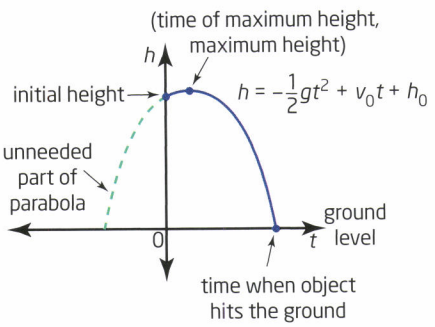
The formula $h = -\frac{1}{2}gt^2 + v_0t + h_0$ can be used to model the height of a projectile, where g is acceleration due to gravity, which is 9.8 m/s^2 on Earth, v_0 is the initial vertical velocity, in metres per second, and h_0 is the initial height, in metres.

a) Create a model for the height of a toy rocket launched upward at 60 m/s from the top of a 3-m platform.

b) How long would the rocket take to fall to Earth, rounded to the nearest hundredth of a second?

c) What is the maximum height of the rocket, rounded to the nearest metre?

d) Over what time interval is the height of the toy rocket greater than 150 m? Round to the nearest hundredth of a second.



Solution

a) Use the formula $h = -\frac{1}{2}gt^2 + v_0t + h_0$ with $g = 9.8$, $v_0 = 60$, and $h_0 = 3$.

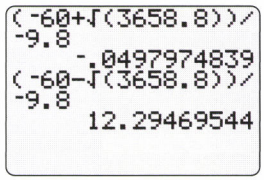
$$h = -\frac{1}{2}(9.8)t^2 + 60t + 3$$
$$= -4.9t^2 + 60t + 3$$

b) Method 1: Use Pencil and Paper

Let $h = 0$ and apply the quadratic formula.

For $-4.9t^2 + 60t + 3 = 0$, $a = -4.9$, $b = 60$, and $c = 3$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-60 \pm \sqrt{60^2 - 4(-4.9)(3)}}{2(-4.9)}$$
$$= \frac{-60 \pm \sqrt{3600 + 58.8}}{-9.8}$$
$$= \frac{-60 \pm \sqrt{3658.8}}{-9.8}$$



So, $t \doteq -0.05$ or $t \doteq 12.29$.

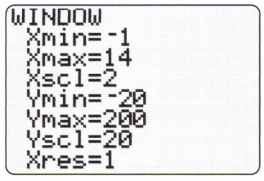
Since t represents the time after the rocket was launched, $t > 0$.

Therefore, reject the root $t = -0.05$.

It would take about 12.29 s for the rocket to fall to Earth.

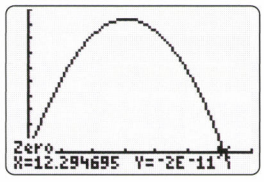
Method 2: Use a Graphing Calculator

Graph the relation $h = -4.9t^2 + 60t + 3$ using the window settings shown.



Then, use the Zero operation of the graphing calculator to find the t-intercept farthest to the right.

It would take about 12.29 s for the rocket to fall to Earth.



c) Method 1: Use Pencil and Paper

The t-coordinate of the vertex can be found using $t = -\frac{b}{2a}$.

$$t = -\frac{60}{2(-4.9)}$$
$$\doteq 6.12$$

Substitute 6.12 for t into $h = -4.9t^2 + 60t + 3$.

$h = -4.9(6.12)^2 + 60(6.12) + 3$

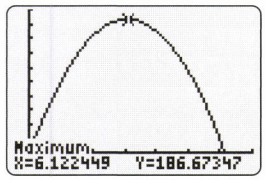
$\doteq 187$

The rocket reached a maximum height of about 187 m.

Method 2: Use a Graphing Calculator

Use the Maximum operation of a graphing calculator to find the coordinates of the vertex.

The rocket reached a maximum height of about 187 m.



d) Method 1: Use Pencil and Paper

Let $h = 150$ and then express the equation in the form $ax^2 + bx + c = 0$.

$-4.9t^2 + 60t + 3 = 150$

$-4.9t^2 + 60t - 147 = 0$

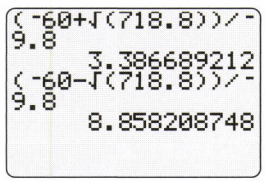
Use the quadratic formula with $a = -4.9$, $b = 60$, and $c = -147$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-60 \pm \sqrt{60^2 - 4(-4.9)(-147)}}{2(-4.9)}$$

$$= \frac{-60 \pm \sqrt{3600 - 2881.2}}{-9.8}$$

$$= \frac{-60 \pm \sqrt{718.8}}{-9.8}$$

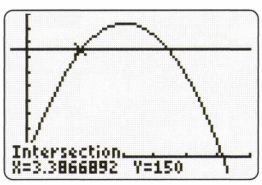


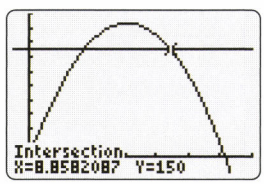
So, $t \doteq 3.39$ or $t \doteq 8.86$.

Therefore, the height of the toy rocket is greater than 150 m from about 3.39 s to 8.86 s.

Method 2: Use a Graphing Calculator

Let $h = 150$. Graph both the original relation $h = -4.9x^2 + 60x + 3$ and $h = 150$. Use the Intersect operation of a graphing calculator to find the t-coordinates of the points of intersection.





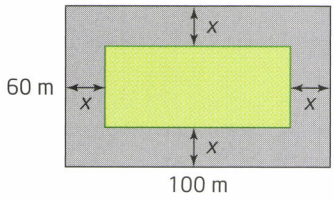
Therefore, the height of the toy rocket is greater than 150 m from about 3.39 s to 8.86 s.

Example 2 Width of a Path

A rectangular park measures 100 m by 60 m. A path of constant width is to be paved around the perimeter. The mayor wants to be sure that the path does not reduce the area of grass by more than 10%. What is the maximum allowable width of the path, rounded to the nearest tenth of a metre?

Solution

Let x represent the width of the path.
Write and solve an equation to find x.
The dimensions of the park inside the path are $100 - 2x$ by $60 - 2x$.
The original area is 100×60, or $6000\ m^2$.
The minimum new area is $90\% \times 6000$, or $5400\ m^2$.



$$(100 - 2x)(60 - 2x) = 5400$$
$$6000 - 200x - 120x + 4x^2 = 5400 \quad \text{Expand the left side.}$$
$$4x^2 - 320x + 6000 = 5400 \quad \text{Simplify.}$$
$$4x^2 - 320x + 600 = 0 \quad \text{Write in the form } ax^2 + bx + c = 0.$$
$$x^2 - 80x + 150 = 0 \quad \text{Divide both sides by 4 to simplify.}$$

Use the quadratic formula with $a = 1$, $b = -80$, and $c = 150$.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$= \frac{-(-80) \pm \sqrt{(-80)^2 - 4(1)(150)}}{2(1)}$$
$$= \frac{80 \pm \sqrt{6400 - 600}}{2}$$
$$= \frac{80 \pm \sqrt{5800}}{2}$$

```
(80+√(5800))/2
        78.07886553
(80-√(5800))/2
        1.921134471
```

So, $x \doteq 78.1$ or $x \doteq 1.9$.
It appears that there are two possible widths for the path, 78.1 m and 1.9 m.

Check.
If the width of the path is 78.1 m, then the dimensions of the grass are $100 - 2(78.1)$ by $60 - 2(78.1)$, or -56.2 m by -96.2 m. This is not possible because the dimensions must be positive.

If the width of the path is 1.9 m, then the dimensions of the grass are $100 - 2(1.9)$ by $60 - 2(1.9)$, or 96.2 m by 56.2 m.

The area of grass is 96.2×56.2, or $5406.44\ m^2$, which is greater than $5400\ m^2$. Therefore, the path should be no wider than 1.9 m.

Example 3 Consecutive Numbers

The product of two consecutive even numbers is 5624.
What are the numbers?

Solution

Consecutive even numbers differ by two.
Let x represent one even number and $x + 2$ represent the other.
Write and solve an equation to find x.

$$x(x + 2) = 5624 \quad \text{Expand the left side.}$$
$$x^2 + 2x = 5624 \quad \text{Simplify.}$$
$$x^2 + 2x - 5624 = 0 \quad \text{Write in the form } ax^2 + bx + c = 0.$$
$$(x - 74)(x + 76) = 0 \quad \text{Factor the left side.}$$
$$x - 74 = 0 \quad \text{or} \quad x + 76 = 0$$
$$x = 74 \quad \text{or} \quad x = -76$$

Verify that each value of x solves the problem.
If $x = 74$, then the next consecutive even number is 76.
$74 \times 76 = 5624$
If $x = -76$, then the next consecutive even number is -74.
$(-76) \times (-74) = 5624$

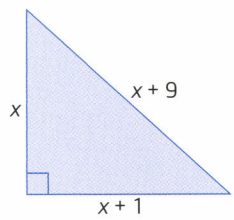
The numbers are 74 and 76 or -76 and -74.

Example 4 Right Triangle

One leg of a right triangle is 1 cm longer than the other leg. The length of the hypotenuse is 9 cm greater than that of the shorter leg. Find the lengths of the three sides.

Solution

Let x represent the length of the shorter leg. The length of the longer leg is $x + 1$ and the length of the hypotenuse is $x + 9$.



Use the Pythagorean theorem.

$$x^2 + (x + 1)^2 = (x + 9)^2$$
$$x^2 + x^2 + 2x + 1 = x^2 + 18x + 81$$
$$2x^2 + 2x + 1 = x^2 + 18x + 81$$
$$x^2 - 16x - 80 = 0$$
$$(x - 20)(x + 4) = 0$$
$$x - 20 = 0 \quad \text{or} \quad x + 4 = 0$$
$$x = 20 \quad \text{or} \quad x = -4$$

Since the length cannot be negative, reject the root $x = -4$.

Check $x = 20$.
Find the side lengths of the triangle.

Shorter leg: 20	Longer leg:	Hypotenuse:
	$x + 1$	$x + 9$
	$= 20 + 1$	$= 20 + 9$
	$= 21$	$= 29$

Verify that this is a right triangle.

$20^2 + 21^2$
$= 400 + 441$
$= 841$
$= 29^2$

Therefore, the three sides of the right triangle have lengths 20 cm, 21 cm, and 29 cm.

Example 5 Path of a Soccer Ball

Matt uses a digital video recorder to record Paul's kick of a soccer ball. He then chooses an origin and a scale, and makes measurements of the height of the soccer ball at several horizontal distances from the television screen during playback.

Horizontal Distance (m)	2.5	5.0	7.5	10.0	12.5	15.0
Height (m)	3.25	4.80	5.75	6.10	5.85	5.00

a) Use technology to determine an equation for the quadratic relation.

b) Where does the soccer ball hit the ground? Round to the nearest tenth of a metre.

Solution

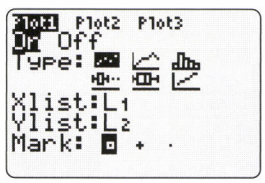
a) Method 1: Use a Graphing Calculator

Clear all lists by pressing [2nd] [MEM], selecting **4:ClrAllLists**, and pressing [ENTER].

Enter the horizontal distance values in list **L1** and the height values in list **L2**.

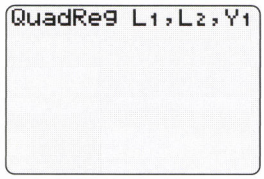
Set up the scatter plot using the settings shown.

Press [ZOOM] and select **9:ZoomStat**.



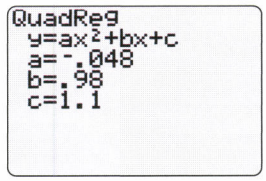
Determine the equation of the curve of best fit.

- Press [STAT], cursor over to display the **CALC** menu, and select **5:QuadReg**.
- Press [2nd] [L1] [,] [2nd] [L2] [,]. Press [VARS], cursor over to display the **Y-VARS** menu, select **1:Function**, and then select **1:Y1**.



• Press ENTER to get the QuadReg screen.

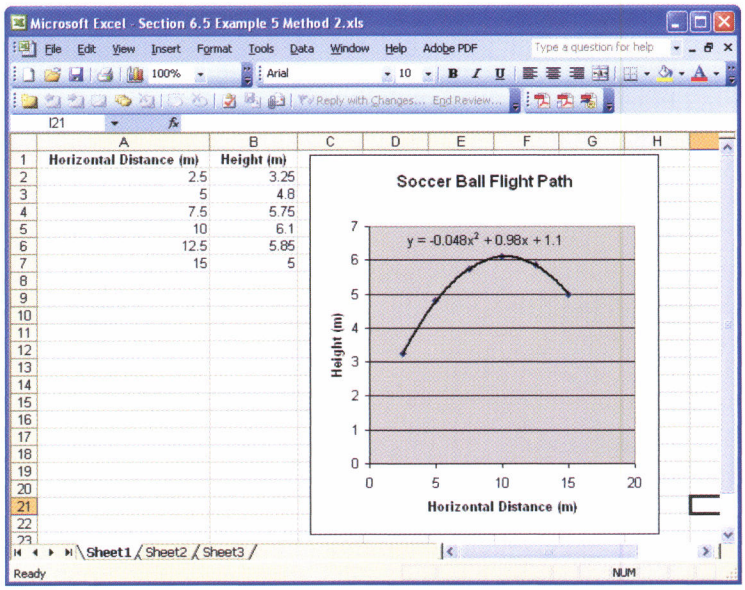
The equation is $y = -0.48x^2 + 0.98x + 1.1$.



Method 2: Use a Spreadsheet

Enter the horizontal distance values in the first column and the height values in the second column. Select the data, and click the chart wizard. For chart type, select scatter. Enter a title and the labels for the axes.

Right-click on one of the data points, and add a trend line to the graph. Select a polynomial regression, of order 2, and the option to display the equation on the graph.



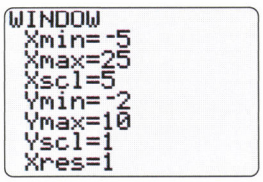
	Horizontal Distance (m)	Height (m)
2	2.5	3.25
3	5	4.8
4	7.5	5.75
5	10	6.1
6	12.5	5.85
7	15	5

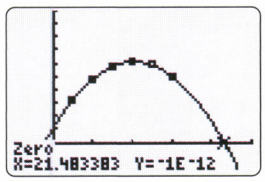
b) Method 1: Use a Graphing Calculator

Use the window settings shown.

Press GRAPH.

Use the Zero operation to find the x-intercept farthest to the right. The soccer ball hits the ground about 21.5 m from where it was kicked.





Method 2: Use a Spreadsheet

Extrapolate the curve until it crosses the horizontal axis. If necessary, adjust the axis parameters to make the graph easier to read. Estimate the value of x where this occurs. The soccer ball hits the ground about 21.5 m from where it was kicked.

Key Concepts

- The solution to some problems can be determined by multiplying two linear expressions together, resulting in a quadratic expression.
- Gravitational pull causes projectiles to have a height-time relationship that is quadratic, modelled by $h = -\frac{1}{2}gt^2 + v_0t + h_0$, where g is the acceleration due to gravity, which is 9.8 m/s^2 on Earth, v_0 is the initial vertical velocity, in metres per second, and h_0 is the initial height, in metres, above the ground.
- Length-area relationships are quadratic since area is a length times a length, measured in square units.
- Follow these steps when determining an algebraic solution:
  - Define your variables.
  - Write an equation to model the situation.
  - Simplify the equation, if necessary.
  - Solve the equation using an appropriate method.
  - Consider the allowable values of the unknown. Reject a solution if necessary, providing an appropriate reason.
  - Provide a concluding statement, answering the original question.

Communicate Your Understanding

C1 In Example 4, a quadratic model is used to solve the problem. Explain why a quadratic equation occurs.

C2 Multiplying two linear expressions together results in a quadratic expression. Explain why.

C3 Fahad's solutions to a quadratic model involving the dimensions of a triangle were 4.5 cm and −3.8 cm. What should his next step be? Explain.

Practise

For help with questions 1 and 2, see Example 1.

1. a) Create a quadratic model for the height of a toy rocket launched upward at 45 m/s from a 2-m platform.

b) How long would the rocket take to fall to Earth, rounded to the nearest hundredth of a second?

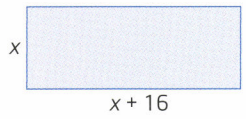
2. A firework is launched upward at an initial velocity of 49 m/s, from a height of 1.5 m above the ground. The height of the firework, in metres, after t seconds, is modelled by the equation $h = -4.9t^2 + 49t + 1.5$.

a) What is the maximum height of the firework above the ground?

b) Over what time interval is the height of the firework greater than 100 m above the ground? Round to the nearest hundredth of a second.

For help with question 3, see Example 2.

3. The length of a rectangle is 16 cm greater than its width. The area is 35 m^2. Find the dimensions of the rectangle, to the nearest hundredth of a metre.



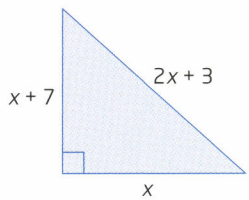
For help with questions 4 and 5, see Example 3.

4. The product of two consecutive numbers is 3306. What are the numbers?

5. Determine two consecutive odd integers whose product is 323.

For help with question 6, see Example 4.

6. The length of one leg of a right triangle is 7 cm more than that of the other leg. The length of the hypotenuse is 3 cm more than double that of the shorter leg. Find the lengths of each of the three sides.



For help with question 7, see Example 5.

7. **Use Technology** Measurements from the flight path of a tennis ball are recorded.

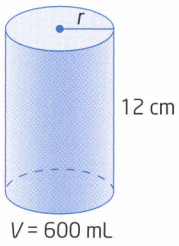
Horizontal Distance (m)	6	8	10	12	14
Height (m)	4.4	4.9	5.0	4.7	4.0

a) Use a graphing calculator or a spreadsheet to create a scatter plot of the data and add a curve of best fit.

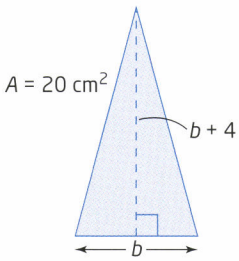
b) Determine the equation of the quadratic relation.

Connect and Apply

8. A cylindrical can with height 12 cm has capacity 600 mL. What is its radius, to the nearest millimetre? [Remember that 1 mL = 1 cm^3.]



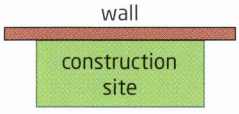
9. The area of a triangle is 20 cm^2, and the altitude is 4 cm greater than the base. Find the length of the base, to the nearest millimetre.



10. The sum of the squares of two consecutive integers is 365. Find the integers.

11. A rectangle has perimeter 23 cm. Its area is 33 cm^2. Determine the dimensions of the rectangle. Include a diagram in your solution.

12. A rectangular construction site is enclosed on three sides using 1200 m of fencing. The remaining side is formed by an existing wall. What dimensions enclose 180 000 m^2 of land?



13. The three sides of a right triangle are consecutive even integers. What is the length of each side?

14. A ladder is 6 m long. If the height of the top of the ladder must be no greater than 10 times the distance from the base to the wall, how high up a wall can the top of the ladder be placed? Include a diagram in your solution. Round to the nearest millimetre.

15. A science experiment involves launching a small rocket. The following measurements are taken:

Initial height: 0.61 m
Initial vertical velocity: 36.85 m/s

a) Create a quadratic model for the height, in metres, of the rocket after a given number of seconds.

b) Verify the following results of the experiment:
Total time in the air: 7.54 s
Maximum height: 69.89 m

c) Sketch a graph of this relation and label the key information as in Example 1 of this section.

16. The acceleration due to gravity on Earth is 9.8 m/s^2. A ball is thrown upward at an initial velocity of 15 m/s from a height of 1 m above the ground. Round answers to the nearest tenth.

a) Write an equation for the height of the ball.

b) What is the height of the ball after 1 s?

c) After how many seconds does the ball land?

d) What is the maximum height of the ball? When does this occur?

e) Repeat parts a) to d) for a ball thrown on the Moon, where $g = 1.62$ m/s^2.

f) Repeat parts a) to d) for a ball thrown on Jupiter, where $g = 23.1$ m/s^2.

17. Sherri sells photos of athletes to baseball, basketball, and hockey fans after their games. Her regular price is \$10 per photograph, and she usually sells about 30 photographs. Sherri finds that, for each reduction in price of \$0.50, she can sell an additional two photographs.

a) Total sales revenue is the product of the number of units sold and the price. Make an algebraic model to represent Sherri's total sales revenue.

b) At what price will Sherri's revenue be \$150?

c) At what price will her maximum revenue occur?

d) At what price will her revenue be \$0?

e) Graph the relationship between revenue and the number of price reductions. Which features on the graph represent the solutions to parts b), c), and d)?

18. A rectangular picture frame measures 20 cm by 30 cm. A new frame is to be made by increasing each side length by the same amount. The resulting enclosed area is to be 1064 cm^2. Find the dimensions of the new picture frame. Include a diagram in your solution.

19. A rectangular garden measures 15 m by 24 m. A larger garden is to be made by increasing each side length by the same amount. The resulting area is to be 1.5 times the original area. Find the dimensions of the new garden, to the nearest tenth of a metre. Include a diagram in your solution.

20. The length of a rectangular field is 2 m greater than three times its width. The area of the field is 1496 m^2. What are the dimensions of the field?

21. An open-topped box is to be made from a rectangular piece of tin measuring 50 cm by 40 cm by cutting squares of equal size from each corner. The base area is to be 875 cm^2.

a) Draw a diagram representing the information.

b) What is the side length of the squares being removed?

c) What is the volume of the box?

22. A photograph measures 21 cm by 15 cm. A strip of constant width is to be cut from each side of the photo, so the area is reduced to 216 cm^2. Find the width of the cut. Include a diagram in your solution.

23. A photograph measures 20 cm by 16 cm. A strip of constant width is to be cut off the top and one side of the photo, so the area is reduced to 60% of the area of the original photo. Find the width of the cut. Include a diagram in your solution.

24. A rectangular field measures 15 m by 20 m. A rectangular area is to be fenced in by reducing each dimension by the same amount. The fenced-in area will be $\frac{1}{2}$ the original area. What will the dimensions of the fenced-in area be? Include a diagram in your solution.

25. A rotating liquid surface takes on the shape of a parabolic mirror. This is the principle behind the 6 m in diameter reflecting telescope at the University of British Columbia, which uses mercury as a reflecting surface. The vertex is 23 cm below the edges. Find an equation to model the parabolic cross section of the mirror. Note: The mirror must always point straight up for the principle to work.

Go to www.mcgrawhill.ca/links/principles10 and follow the links to learn more about liquid mirror telescopes.

26. Use Technology An automotive magazine tested the stopping distance required by a particular car, starting from various speeds. The data are shown in the table.

Speed (km/h)	Stopping Distance (m)
30	3.6
40	6.2
50	10.0
60	14.1
70	19.6
80	27.8
90	36.5
100	49.3

a) Use technology to create a scatter plot of the data and add a curve of best fit.

b) Determine the equation of the quadratic relation.

c) Extrapolate to determine the stopping distance for a speed of 110 km/h.

d) Determine what approximate speed results in stopping distances of 30 m, 65 m, and 200 m.

e) Comment on the validity of extrapolation for this model.

Extend

27. Determine the number of points of intersection of each pair of parabolas. Justify your answer.

a) $y = x^2 + 2x + 7$
$y = x^2 - 4x - 1$

b) $y = 3x^2 - 12x + 16$
$y = -2x^2 - 4x + 3$

c) $y = x^2 - 6x + 10$
$y = 5x^2 - 30x + 46$

28. Use Technology Wind chill is the temperature that the air feels like at a given wind speed. The chart shows the wind chill, in degrees Celsius, for various air temperatures and wind speeds.

Wind Speed (km/h)	Air Temperature (°C)					
	20	15	10	5	0	−5
10	17	12	7	2	−3	−8
20	14	9	4	−1	−6	−11
30	11	6	1	−4	−9	−14
40	9	4	−1	−6	−11	−16
50	8	3	−2	−7	−12	−17
60	7	2	−3	−8	−13	−18

a) Use a graphing calculator to create a quadratic model for wind chill at a given wind speed, for an air temperature of 5°C.

b) Use a graphing calculator to create a quadratic model for wind chill at a given air temperature, for a wind speed of 60 km/h.

c) How accurate are your wind-chill models for 5°C at 60 km/h? Explain any differences in the results.

29. Explain why the quadratic model for the height of a projectile changes if the initial speed is measured in a diagonal direction instead of vertically.

30. In the TV show *Junkyard Wars*, a trebuchet was used to catapult a pumpkin from a height of 4 m for a total horizontal distance of 24 m. It reached a maximum height of 14 m. At what horizontal distances was the height of the pumpkin 10 m, to the nearest metre?

31. Math Contest The French mathematician Jacques Binet (1786−1856) developed a formula for the terms in the Fibonacci sequence:

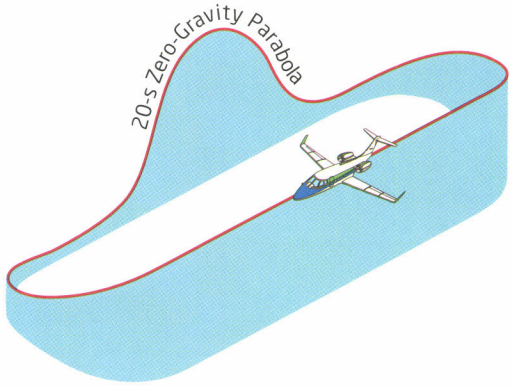
$$f_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}}$$

Verify that Binet's formula gives the correct values for the first five terms of the Fibonacci sequence.

Making Connections

Parabolic Flight

The Falcon 20 aircraft of the National Research Council is a modified commercial jet that can create conditions similar to zero gravity by performing parabolic manoeuvres. During parabolic flight, the aircraft climbs into a steep arc and then descends. The aircraft is in free fall for 15 s to 20 s, which creates the zero-gravity conditions. During this time, Canadian space science researchers can conduct experiments.



Chapter 6 Review

6.1 Maxima and Minima, pages 264–273

1. Rewrite each relation in the form $y = a(x - h)^2 + k$ by completing the square. Use algebra tiles or a diagram to support your solution.

a) $y = x^2 + 8x - 7$

b) $y = x^2 + 2x + 7$

c) $y = x^2 + 4x + 6$

d) $y = x^2 + 6x - 3$

2. Find the vertex of each parabola. Sketch the graph, labelling the vertex, the axis of symmetry, and two other points.

a) $y = x^2 + 12x + 30$

b) $y = x^2 - 14x + 50$

c) $y = -x^2 + 6x - 7$

d) $y = 5x^2 - 40x + 76$

3. **Use Technology** Use a graphing calculator to find the maximum or minimum point of each parabola, rounded to the nearest tenth.

a) $y = 2.1x^2 - 2.5x + 8.0$

b) $y = -338x^2 + 127x - 212$

c) $y = \frac{1}{5}x^2 - \frac{2}{25}x + \frac{3}{5}$

6.2 Solve Quadratic Equations, pages 274–281

4. Solve by factoring. Check your solutions.

a) $x^2 + 10x + 21 = 0$

b) $m^2 + 8m - 20 = 0$

c) $6y^2 + 21y + 9 = 0$

d) $5n^2 + 13n - 6 = 0$

5. Solve.

a) $y^2 = 8y + 9$

b) $x^2 - 8x = -7$

c) $3m^2 = -10m - 7$

d) $30x - 25x^2 = 9$

e) $8k^2 = -5 + 14k$

f) $3x^2 + 2 = -5x$

6. The length of the hypotenuse of a right triangle is 1 cm more than triple that of the shorter leg. The length of the longer leg is 1 cm less than triple that of the shorter leg. Find the lengths of the three sides of the triangle.

6.3 Graph Quadratics Using the *x*-Intercepts, pages 282–291

7. Find the *x*-intercepts and the vertex of each quadratic relation. Then, sketch its graph.

a) $y = x^2 + 8x + 12$

b) $y = x^2 - 4x - 5$

c) $y = -x^2 - 6x + 27$

d) $y = 3x^2 + 10x + 8$

e) $y = -x^2 - 3x$

f) $y = x^2 - 4$

8. If two different quadratic relations have the same zeros, will they necessarily have the same axis of symmetry? Will they have the same vertex? Explain your reasoning.

9. Find the value(s) of k so that each parabola has only one *x*-intercept.

a) $y = x^2 + kx + 100$

b) $y = -4x^2 + 28x + k$

6.4 The Quadratic Formula, pages 292–303

10. Use the quadratic formula to solve each equation. Express your answers as exact roots.

a) $-3x^2 - 2x + 5 = 0$

b) $9x^2 - 8x - 3 = 0$

c) $5x^2 + 7x + 1 = 0$

d) $25x^2 + 90x + 81 = 0$

11. If a baseball is batted at an angle of 35° to the ground, the distance the ball travels can be estimated using the equation $d = 0.0034s^2 + 0.004s - 0.3$, where s is the bat speed, in kilometres per hour, and d is the distance flown, in metres. At what speed does the batter need to hit the ball in order to have a home run where the ball flies 125 m? Round to the nearest tenth.

6.5 Solve Problems Using Quadratic Equations, pages 304–315

12. A diver bounces off a 3-m springboard at an initial upward speed of 4 m/s.

a) Create a quadratic model for the height of the diver above the water.

b) After how many seconds does the diver enter the water? Round to the nearest hundredth of a second.

c) Over what time interval is the height of the diver greater than 3.5 m above the water? Round to the nearest hundredth of a second.

13. A rectangular garden measures 5 m by 7 m. Both dimensions are to be extended by the same amount so that the area of the garden is doubled. By how much should the dimensions increase, to the nearest tenth of a metre?

14. The Sticker Warehouse sells rolls of stickers for \$4.00 each. The average customer buys six rolls of stickers. The owner finds that, for every \$0.25 decrease in price, the average customer buys an additional roll.

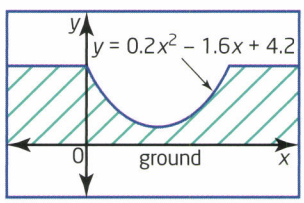
a) Total sales revenue is the product of the number of units sold and the price. Make an algebraic model to represent The Sticker Warehouse's total revenue per customer.

b) With how many price reductions will the revenue per customer be \$30?

c) What is the maximum predicted sales revenue per customer? With how many price reductions will this occur?

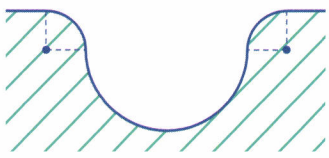
Chapter Problem Wrap-Up

In Section 6.1, question 18, and Section 6.4, question 12, you explored the dimensions of a parabolic skateboard half-pipe ramp defined by the equation $y = 0.2x^2 - 1.6x + 4.2$, where x is the horizontal distance, in metres, from the platform, and y is the height, in metres, above the ground.



a) The company's standard model of a half-pipe ramp with a semicircular cross section is modelled by the equation $x^2 + y^2 = 16$, where x and y are the horizontal and vertical distances, in metres, respectively, from the centre of the semicircle. Write a one-paragraph report comparing the two half-pipe ramps.

b) The company is considering a deluxe ramp that includes two inverted quarter circles at the top of the standard model of a half-pipe ramp, as shown. The radius of the quarter circles is half the radius of the half-pipe. Add to your report any advantages or disadvantages that you see in this new design.



Chapter 6 Practice Test

1. Graph each parabola by completing the square. Label the vertex, the axis of symmetry, and two other points.

a) $y = x^2 + 6x + 4$

b) $y = -x^2 + 8x - 3$

c) $y = 3x^2 + 24x + 10$

2. Solve each quadratic equation by factoring.

a) $x^2 - 5x + 4 = 0$ **b)** $9y^2 - 1 = 0$

c) $x^2 = 3x + 10$ **d)** $9b^2 - 12b + 4 = 0$

e) $3x^2 + 13x = 10$ **f)** $6m^2 + 30m = 0$

g) $5x^2 = 10x$ **h)** $4d + 1 = -4d^2$

3. Explain the process for finding the vertex of a quadratic relation using factoring. Include an example in your explanation.

4. Find the x-intercepts, axis of symmetry, and vertex of each parabola. Then, graph the relation, labelling it fully.

a) $y = x^2 + 2x - 35$

b) $y = -x^2 - x + 20$

c) $y = -3x^2 - 6x$

d) $y = x^2 - 10x + 25$

5. Use the quadratic formula to solve, if possible. Express your answers as exact roots.

a) $4x^2 - 11x - 3 = 0$ **b)** $x^2 + 5x = 7$

c) $9x^2 = 30x - 25$ **d)** $7k^2 - 9k + 3 = 0$

e) $4s^2 - 9s = -3$ **f)** $3t^2 - 7 = t$

6. Use an appropriate method to find the roots of each equation.

a) $3x^2 + 12x + 6 = 0$

b) $x^2 - 8x + 3 = 0$

c) $4m^2 - 10 = 0$

d) $-5x^2 + 10x = 5$

e) $(k - 5)^2 = 16$

f) $\frac{x^2}{2} + x + \frac{1}{2} = 0$

g) $2(m - 1)^2 = (m + 2)(m + 1)$

h) $(5x + 2)(3x - 1) = 4x^2 + 5$

7. Compare the axis of symmetry of $y = 3x^2 + 18x - 17$ to that of $y = 3x^2 + 18x + 1$. Explain your findings.

8. The path of a firework is modelled using the equation $h = -5d^2 + 20d + 1$, where h is the height, in metres, above the ground and d is the horizontal distance, in metres. What is the maximum height of the firework?

9. A parabola is defined by the equation $y = -2(x + 1)^2 + 18$.

a) Without solving, explain how you can tell how many x-intercepts there are.

b) Given the current form of the equation, what is the easiest way to find the x-intercepts?

c) How far apart are the x-intercepts?

10. Write a quadratic equation in the form $ax^2 + bx + c = 0$ for each situation, where a, b, and c are integers.

a) The roots of the equation are 5 and -3.

b) The roots of the equation are $\frac{1}{2}$ and $\frac{3}{5}$.

11. An equipment storage shed has a parabolic cross section modelled by the relation $h = -d^2 + 4d$, where h is the height, in metres, and d is the horizontal distance, in metres, from one edge of the shed.

a) How wide and how tall is the shed?

b) Sketch the graph.

c) For what values of d is the relation valid? Explain.

12. The cost, in dollars, of operating a machine per day is given by the formula $C = 3t^2 - 96t + 1014$, where t is the time the machine operates, in hours. What is the minimum cost of running the machine? For how many hours must the machine run to reach this minimum cost?

13. A triangle has base $2x + 1$ and height $6x - 3$. What value of x would give an area of 240 m^2? Round to the nearest hundredth.

14. The relation $d = 0.0052s^2 + 0.13s$ models the stopping distance, d, in metres, of a car travelling at a speed of s, in kilometres per hour, when the driver brakes hard. At what speed was a car travelling if its stopping distance is 20 m? Round to the nearest tenth.

15. Van dives off a 4-m springboard. His height, h, in metres, above the surface of the water is defined by the relation $h = -d^2 + 3d + 4$, where d is his horizontal distance, in metres, from the end of the board.

a) Determine the zeros of the relation.

b) Sketch a graph of the relation.

c) For what values of d is the relation valid?

d) What is Van's horizontal distance from the board when he enters the water?

e) What is Van's maximum height above the water?

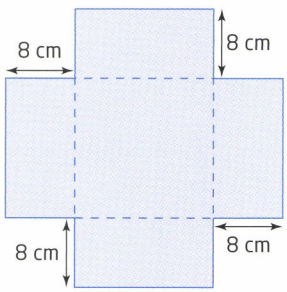
16. In a volleyball match, Jenny serves the volleyball at 14 m/s, from a height of 2.5 m above the court. The height of the ball in flight can be estimated using the equation $h = -4.9t^2 + 14t + 2.5$, where h is the height, in metres, and t is the time, in seconds, after she serves the ball.

a) What is the maximum height of the volleyball above the court? When does this occur? Round answers to the nearest tenth.

b) If a player on the other team contacts the ball at a height of 0.5 m above the court, how long does it take for the ball to reach her? Round to the nearest second.

17. The hypotenuse of a right triangle has length 17 cm. The sum of the lengths of the legs is 23 cm. What are their lengths?

18. An open-topped box is to be constructed from a square piece of cardboard by removing a square with side length 8 cm from each corner and folding up the edges. The resulting box is to have a volume of 512 cm^3. Find the dimensions of the original piece of cardboard.



Achievement Check

19. a) Solve the quadratic equation $x^2 - 2x - 15 = 0$ by factoring and by graphing. Use the graph to find the minimum value of the quadratic relation $y = x^2 - 2x - 15$.

b) Solve the quadratic equation $x^2 + 2x - 15 = 0$ by completing the square and by using a computer algebra system.

c) For what value of b will the quadratic equation $9x^2 + bx + 25 = 0$ have one root? What is the value of the root?

d) For what values of b will the quadratic equation $x^2 + bx + 10 = 0$ have *no* real roots?

Chapters 4 to 6 Review

Chapter 4 Quadratic Relations

1. Use finite differences to determine whether each relation is linear, quadratic, or neither.

a)

x	y
-2	9
-1	7
0	5
1	3
2	1

b)

x	y
-2	-3
-1	3
0	5
1	3
2	-3

2. Sketch the graph of each quadratic relation. Describe the transformation from the graph of $y = x^2$.

a) $y = x^2 + 2$

b) $y = (x + 3)^2$

c) $y = -\frac{1}{4}x^2$

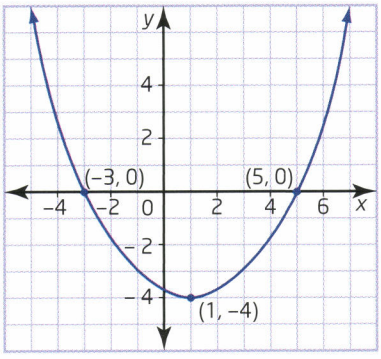
3. Dianne dove from the 10-m diving board. Her height h, in metres, above the water when she is x metres away from the end of the board is given by $h = -(x - 1)^2 + 11$.

a) Sketch a graph of her dive.

b) What was her maximum height above the water?

c) What horizontal distance had she travelled when she entered the water? Answer to the nearest tenth of a metre.

4. Determine an equation to represent the parabola in the form $y = a(x - r)(x - s)$.



5. Evaluate.

a) 2^{-4}

b) $(-3)^{-2}$

c) 25^0

d) 8^{-1}

e) $(-1)^{12}$

f) $\left(\frac{3}{4}\right)^{-3}$

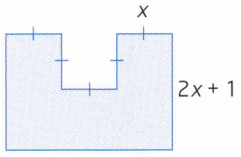
6. Cobalt-60 is a radioactive element that is used to sterilize medical equipment. Cobalt-60 decays to $\frac{1}{2}$, or 2^{-1}, of its original amount after every 5.2 years. Determine the remaining mass of 20 g of cobalt-60 after

a) 20.8 years

b) 36.4 years

Chapter 5 Quadratic Expressions

7. Write an algebraic expression to represent the area of the figure. Expand and simplify.



8. Expand and simplify.

a) $(n + 3)(n - 3)$

b) $(h + 5)^2$

c) $(d - 4)(d - 2)$

d) $(m + 3)(m + 7)$

e) $(3t - 5)(3t + 5)$

f) $(x - 7)^2$

9. Expand and simplify.

a) $x(3x + 1)(2x - 5)$

b) $(2k + 3)^2 - (k + 2)(k - 2)$

c) $5(y - 4)(3y + 1) + (3y - 4)^2$

d) $3(2a + 3b)(3a - 2b)$

10. The area of a rectangle is given by the expression $8x^2 + 4x$. Draw diagrams to show the possible rectangles, labelling the length and width of each.

11. Factor, if possible.

a) $y^2 + 12y + 27$

b) $x^2 + 2x - 3$

c) $n^2 + 22n + 21$

d) $p^2 - 8p + 15$

e) $x^2 + 2x - 15$

f) $k^2 - 5k + 24$

12. Factor.

a) $p^2 + 12p + 36$

b) $9d^2 - 6d + 1$

c) $x^2 - 49$

d) $4a^2 - 20a + 25$

e) $8t^2 - 18$

f) $a^2 - 4b^2$

13. Find the value of k so that each trinomial can be factored over the integers.

a) $m^2 + km + 10$

b) $9a^2 - ka + 4$

14. The area of a circle is given by the expression $\pi(4x^2 + 36x + 81)$. What expression represents the diameter of this circle?

Chapter 6 Quadratic Equations

15. Complete the square to rewrite each relation in the form $y = a(x - h)^2 + k$. Then, sketch a graph of each parabola, labelling the vertex, the axis of symmetry, and two other points.

a) $y = x^2 + 6x - 16$

b) $y = x^2 - 8x + 7$

c) $y = x^2 + 4x + 10$

d) $y = -x^2 + 6x - 8$

e) $y = 2x^2 + 8x + 5$

f) $y = -2x^2 - 12x - 7$

16. Solve by factoring. Check your solutions.

a) $x^2 - 14x + 24 = 0$

b) $n^2 + 4n - 21 = 0$

c) $m^2 - 16 = 0$

d) $2y^2 + 5y + 2 = 0$

e) $7t^2 = 70t - 175$

17. Find the x-intercepts and the vertex of each parabola. Then, sketch its graph.

a) $y = x^2 - 3x + 2$

b) $y = x^2 - 16$

c) $y = 2x^2 - 5x - 12$

d) $y = -x^2 - 7x - 12$

e) $y = -3x^2 + 6x$

18. A quadratic relation has roots 0 and -6 and a maximum at $(-3, 4)$. Determine the equation of the relation.

19. Find the exact roots of each equation.

a) $x^2 - 6x + 1 = 0$

b) $3x^2 - 5x + 1 = 0$

c) $3.2x^2 - 5.6x - 7.1 = 0$

d) $2x^2 + 5x = 9$

e) $3x^2 - 8x = -3$

20. The perimeter of a rectangle is 8 m and its area is 2 m^2. Find the length and width of the rectangle to the nearest tenth of a metre.

21. A ferry operator takes tourists to an island. The operator carries an average of 500 people per day for a round-trip fare of \$20. The operator estimates that for each \$1 increase in fare, 20 fewer people will take the trip. What fare will maximize the number of people taking the ferry?

Tasks

Cari Sports Centre

Some of the costs of running Cari Sports Centre decrease as the number of people using the facility increases. Other costs, such as the salaries of staff needed to serve the additional people, grow.

As part of a budget review, city council would like to improve the profit from the Cari Sports Centre. City council surveys people using the centre and develops a formula by examining

- the price of admission to the centre
- the number of tickets sold
- the constant daily expenses in running the centre

The centre manager, Emily, was given the formula that related the number of admission tickets, the price of a ticket, and the daily profit. Unfortunately, Emily lost the paper on which the formula was written, but she had already calculated the following data list.

Ticket Price ($)	Daily Profit ($)
8	56
10	200
12	296
14	344

a) Copy the table and find the first and second differences. What can you conclude about the formula?

b) Make a scatter plot of the data. Draw a curve of best fit. What ticket price will give the greatest daily profit?

c) What are the constant daily expenses?

d) What range of admission prices will ensure that the daily profit is at least $326?

York Leisure Centre

A formula based on admission revenue and maintenance costs is used to determine the daily profit at the York Leisure Centre. The formula expresses the daily profit, y, in dollars, in relation to the price of admission, x, in dollars.

$y = -4x^2 + 100x - 500$

a) Graph this relation. Find the coordinates of the vertex and the equation of the axis of symmetry.

b) Write the relation in the form $y = a(x - h)^2 + k$.

c) What information does the solution of the equation $0 = -4x^2 + 100x - 500$ give about the York Leisure Centre's daily profit situation?

d) If each person entering the York Leisure Centre spends an average of \$2.50 at the coffee shop, how is the rule for daily profit affected? What price should be charged for an admission ticket in this case? Justify your answer.

Abbey Leisure Centre

Abbey Leisure Centre currently has 300 members. The centre's management board wants to expand the facilities. The board has decided to raise the cost of annual membership from the current price of \$400. A survey has found that for each \$20 increase in the annual fee, 10 members will not renew their membership.

Prepare a report for the board of Abbey Leisure Centre, advising them of the cost of annual membership that will maximize the revenue.